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Aspects of *p*-adic geometry: Lipschitz extensions and monotonicity

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor in Science

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Aspects of *p*-adic geometry:
Lipschitz extensions and monotonicity

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Abstract

In this thesis we present the results of four years of research on some aspects of p-adic geometry. A first result is on definable Lipschitz extensions of p-adic functions, a second result is on differentiation in P-minimal structures. Both results will appear in the form of an article in a peer-reviewed mathematical journal.

Firstly, we prove a definable version of Kirszbraun's theorem in a non-Archimedean setting for definable families of functions in one variable. More precisely, let K be a finite field extension of $\mathbb{Q}_p$, then we prove that every definable function $f : X \times Y \to K^s$, where $X \subset K$ and $Y \subset K^r$, that is λ-Lipschitz in the first variable, extends to a definable function $\tilde{f} : K \times Y \to K^s$ that is λ-Lipschitz in the first variable.

Secondly, we prove a p-adic, local version of the Monotonicity Theorem for P-minimal structures. The existence of such a theorem was originally conjectured by Haskell and Macpherson. We approach the problem by considering the first order strict derivative. In particular, we show that, for a wide class of P-minimal structures, the definable functions $f : K \to K$ are almost everywhere strictly differentiable and satisfy the Local Jacobian Property.

The basic facts of p-adic fields are reviewed in the first chapter of this thesis. Model theory and its applications are reviewed in the second chapter. The third chapter contains the new results on definable Lipschitz extensions of p-adic functions, and the forth chapter those on differentiation in P-minimal structures. Finally, the fifth chapter contains a discussion of the main results in this thesis, together with a look at future research.

Samenvatting

Deze thesis is een samenvatting van vier jaar wiskundig onderzoek naar aspecten van p-adische meetkunde. Een eerste resultaat dat we presenteren is een resultaat over definieerbare Lipschitz-uitbreidingen van p-adische functies. Een tweede resultaat gaat over afleidbaarheid in P-minimale structuren. Beide resultaten zullen verschijnen in de vorm van een wetenschappelijk artikel in een wiskundig tijdschrift.

Ten eerste bewijzen we een definieerbare versie van Kirszbraun's stelling in een niet-Archimedische context voor definieerbare families van functies in één variabele. Zij K een eindige velduitbreiding van $\mathbb{Q}_p$, dan bewijzen we meer bepaald dat elke definieerbare functie $f : X \times Y \to K^s$, waarbij $X \subset K$ en $Y \subset K^r$, die λ-Lipschitz is in de eerste variabele, uitgebreid kan worden tot een definieerbare functie $\tilde{f} : K \times Y \to K^s$ die opnieuw λ-Lipschitz is in de eerste variabele.

Ten tweede bewijzen we een p-adische, lokale versie van de Monotoniciteitsstelling voor P-minimale structuren. Het bestaan van deze stelling werd oorspronkelijk geformuleerd als een vermoeden door Haskell en Macpherson. We benaderen dit probleem door eerst te kijken naar de eerste orde strikte afgeleide. In het bijzonder tonen we voor een grote klasse van P-minimale structuren aan dat definieerbare functies $f : K \to K$ bijna overal strikt afleidbaar zijn en voldoen aan de Lokale Jacobiaan-eigenschap.

In het eerste hoofdstuk geven we een overzicht van enkele elementaire eigenschappen van p-adische velden. In het tweede hoofdstuk komt modeltheorie aan bod waarvan we enkele toepassingen bekijken. Het derde hoofdstuk bevat nieuwe resultaten over definieerbare Lipschitz-

uitbreidingen van p-adische functies. Het vierde hoofdstuk bevat nieuwe resultaten over afleidbaarheid in P-minimale structuren. Tenslotte bevat het vijfde hoofdstuk een discussie over alle resultaten uit deze thesis en wordt er een blik geworpen op toekomstig onderzoek.

Deze thesis eindigt met een hoofdstuk voor de niet-wiskundige, waar de lezer wordt meegenomen in een verhaal over een nieuw soort afstand. We laten de lezer kennismaken met de p-adische afstand, die de hoofdrol speelt in alles wat te maken heeft met p-adische meetkunde. Met andere woorden is het eindpunt van het laatste hoofdstuk het beginpunt van het eerste: de cirkel is rond!

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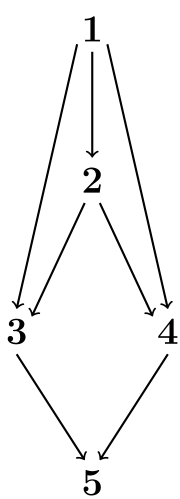


Introduction

In this thesis, the main results are presented that I obtained in the two research projects that I worked on during my PhD. Both projects lie in the same mathematical area, namely that of model theory, applied to p-adic geometry.

The first research project, which was the main project of my PhD, was on definable Lipschitz extensions of p-adic functions. More specifically, the aim of this project was to answer the following question:

Question. Let K be a finite field extension of $\mathbb{Q}_p$ and let $f : S \subset K^r \to K^s$ be a definable and λ-Lipschitz function. Does there exist a definable and λ-Lipschitz function $\tilde{f} : K^r \to K^s$ that agrees with f on S, i.e. $\tilde{f}|_S = f$?

This project originated from a recent result that gave a positive answer to this question in the *real* context. We couldn't answer the question in full generality, but proved the result for definable families of functions in one variable. The strategy to obtain this result was to apply a refined form of p-adic cell decomposition to the domain S of f. On each of these cells we then extended f in an elementary way and finally, we proved that these extensions could be *glued* to obtain the desired extension of f. The results from this research project will be published in the Mathematical Logic Quarterly (see [28]).

The second research project, which I worked on together with Eva Leenknegt, started as a small side project of the first one, but quickly grew to become a full sized project on its own. In this project we tried to answer the following question, which was formulated by Haskell and Macpherson more than fifteen years ago:

Question. Let K be a finite field extension of $\mathbb{Q}_p$ and let $f : S \subset K \to K$ be a function definable in a P-minimal structure. Does there exist definable disjoint subsets U, V of S, with $S \setminus (U \cup V)$ finite, such that $f\big|_U$ is locally constant and $f\big|_V$ is locally strictly monotone?

Again, this question was inspired by the real case, where even a global version holds: the Monotonicity Theorem for o-minimal structures. In this research project, we proved a positive answer to this question for a wide class of P-minimal structures, namely P-minimal structures that are elementary equivalent to a finite field extension of $\mathbb{Q}_p$, which we call *strictly P-minimal structures*. We obtained this result by proving the Local Jacobian Property for strictly P-minimal structures. The results from this research project will be published in the Journal of Symbolic Logic (see [29]).

We end this introduction with a brief overview of the content of the chapters in this thesis, in order of appearance. Chapter 1 and 2 are introductory and provide the machinery that is used in both research projects. Chapter 3 is about definable Lipschitz extensions of p-adic functions, the first research project. Chapter 4 deals with differentiability results in P-minimal structures, the second research project. Chapter 5 contains a general discussion about all the results in this thesis, together with some new results and some thoughts for future research. The last chapter is meant for the non-mathematician and doesn't contain any new results.

In the first chapter, the basic facts of finite field extensions of $\mathbb{Q}_p$ are reviewed (these fields will later on always be called *p-adic fields*). The approach is somewhat unconventional, illustrating basic geometrical properties of p-adic fields using interesting pictures. This approach is chosen since these pictures seem to be of great aid for constructing definable Lipschitz extensions of p-adic functions.

In the second chapter, we introduce the basic notions from model theory along with its applications to p-adic geometry. We provide plenty of examples, sketch the context we will later work in, and demonstrate the main techniques that will be used. We collect some elementary, number theoretical results in view of later use and give proofs where necessary. Finally we conclude with some notions from tame geometry, such as monotonicity and the Jacobian Property.

The third chapter contains the new results on definable Lipschitz extensions of p-adic functions. It is an expansion of the article that will be published in the Mathematical Logic Quarterly (see [28]), in the sense that we go deeper into some proofs of the results and add some pictures to increase clarity.

The forth chapter contains the new results on differentiation in P-minimal structures. This chapter is a direct copy of the article that will be published in the Journal of Symbolic Logic (see [29]).

In the fifth chapter we look back at the main results in this thesis and we also take a look at future research. We point out in which way we believe that the main results could be ameliorated and while doing so, we already obtain a generalization and simplification of the results from chapter 3 on definable Lipschitz extensions of p-adic functions.

Finally, in the last chapter we provide a very basic introduction in p-adic geometry for the non-mathematician. We introduce the reader in the world of the p-adic distance, which is the starting point of the first chapter: the cycle is complete.

Chapter 1

Introduction to *p*-adic geometry

Throughout this thesis we will work with the field of p-adic numbers, denoted by $\mathbb{Q}_p$, and finite field extensions of $\mathbb{Q}_p$, called *p-adic fields*. Entire books have been written about p-adic numbers and p-adic fields, with none of whom we wish to compete. Since the list of interesting things one could say about this subject is longer than the intended length of this *entire thesis*, most properties of the p-adic numbers and p-adic fields will *not* be mentioned. We justify the selection of properties that we *do* mention by the argument that those properties will be in particular of interest to us, for understanding the following chapters of this thesis.

In the first section of this chapter, we overview the basic properties of the p-adic numbers. In the second section we focus on p-adic fields.

1.1 *p*-Adic numbers

The goal of this section is to give some intuition about p-adic, non-Archimedean geometry, together with some images to keep in mind. We use [27] as our main reference, from which we recall some basic definitions and facts. Other interesting references to p-adic numbers are [22] and [25]. First, we give a brief overview of the different possible *constructions* of the p-adic numbers, then we list some useful *basic properties*, and finally, we study some interesting *subsets* of the p-adic numbers.

1.1.1 The construction of $\mathbb{Q}_p$

Throughout this thesis, p will always denote a prime number. There are different ways of constructing the field of p-adic numbers $\mathbb{Q}_p$, which we'll discuss briefly.

The following is the algebraic construction of $\mathbb{Q}_p$. Let $\mathbb{Z}/p^n\mathbb{Z}$ be the ring of integers modulo p^n, and let φ_n be the map

$$\varphi_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z} : x \mod p^n \mapsto x \mod p^{n-1},$$

for $n > 1$. Then one defines the ring of *p-adic integers* $\mathbb{Z}_p$ as the projective limit of the following projective system:

$$\mathbb{Z}/p\mathbb{Z} \xleftarrow{\varphi_2} \mathbb{Z}/p^2\mathbb{Z} \xleftarrow{\varphi_3} \cdots \xleftarrow{\varphi_n} \mathbb{Z}/p^n\mathbb{Z} \xleftarrow{\varphi_{n+1}} \cdots,$$

i.e. $\mathbb{Z}_p$ is by definition equal to

$$\mathbb{Z}_p = \{(x_n)_{n\geq 1} \mid x_n \in \mathbb{Z}/p^n\mathbb{Z}, \text{ and for all } n > 1 : \varphi_n(x_n) = x_{n-1}\}.$$

The ring $\mathbb{Z}_p$ is an integral domain, and one defines the *field of p-adic numbers* $\mathbb{Q}_p$ as the fraction field of $\mathbb{Z}_p$.

There is also the analytic construction of $\mathbb{Q}_p$, which mimics the construction of the real numbers as a *complete* field extension of the rational numbers. We first discuss this construction of $\mathbb{R}$.

Let $|\cdot|$ be the traditional absolute value on $\mathbb{Q}$, i.e. $|x| = x$ if x is positive, $|x| = -x$ if x is negative, and $|0| = 0$. This absolute value induces the traditional distance, i.e. the distance between two rational numbers x and y is equal to $|x - y|$. Say a sequence $(x_n)_{n\geq 1}$ of rational numbers is *Cauchy* if the distance between elements in the tail of the sequence becomes arbitrarily small, or, more formally, if for every $\epsilon > 0$ there exists a natural number N such that for all $i, j > N$, one has $|x_i - x_j| \leq \epsilon$. Not every Cauchy sequence of rational numbers has a limit in $\mathbb{Q}$. For example, let $(x_n)_{n\geq 1}$ be the sequence

$$(1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142135, 1.41421356, \ldots),$$

then clearly this sequence is Cauchy, but $\lim_{n\to\infty} x_n = \sqrt{2}$. The fact that in $\mathbb{Q}$ not every Cauchy sequence has a limit, makes that $\mathbb{Q}$ is not *complete.*

We now want to construct a complete field containing $\mathbb{Q}$. The idea is to consider the set of Cauchy sequences, which contains $\mathbb{Q}$ via the following identification: every element $x \in \mathbb{Q}$ can be viewed as the Cauchy sequence $(x, x, x, \ldots)$. This idea is somewhat too naive, though, since the set of Cauchy sequences of rational numbers is not a field (addition and multiplication on Cauchy sequences are defined componentwise, and obviously not every nonzero Cauchy sequence has a multiplicative inverse). That's why one looks at the equivalence classes of the set of rational Cauchy sequences, with respect to the following equivalence relation: say $(x_n)_{n\geq 1} \sim (y_n)_{n\geq 1}$ if and only if $\lim_{n\to\infty}|x_n - y_n| = 0$. Then the set of all rational Cauchy sequences modulo $\sim$, turns out to be a complete field, containing $\mathbb{Q}$ via the injective map sending $x \in \mathbb{Q}$ to the class of $(x, x, x, \ldots)$. This field can be taken as the definition of $\mathbb{R}$, the field of real numbers, and this notion coincides with all other known constructions of $\mathbb{R}$.

One can mimic this construction of $\mathbb{R}$ to construct $\mathbb{Q}_p$, where nothing is changed except the distance one works with. Let x be any nonzero rational number, then x can be written in the form $x = p^l \frac{a}{b}$, where a and b are coprime integers, $b \neq 0$, and neither a nor b is divisible by p. One then defines $|x|_p = p^{-l}$ and one extends the definition to all rational numbers by putting $|0|_p = 0$. With this definition, $|\cdot|_p$ forms an *absolute value* on $\mathbb{Q}$, i.e. the following three conditions are satisfied:

1. $|x|_p \geq 0$ for all $x \in \mathbb{Q}$ and $|x|_p = 0$ if and only if $x = 0$;
2. $|xy|_p = |x|_p|y|_p$ for all $x, y \in \mathbb{Q}$;
3. $|x - y|_p \leq |x|_p + |y|_p$ for all $x, y \in \mathbb{Q}$.

The third property is called the *triangle inequality.* Every absolute value $|\cdot|$ induces a distance (or metric), by putting $d(x, y) = |x - y|$. The distance that is induced by $|\cdot|_p$ on $\mathbb{Q}$ is called the *p-adic distance.* If one replaces in the construction of $\mathbb{R}$, as described above, all occurrences of $|\cdot|$ by $|\cdot|_p$, one ends up with a complete field containing $\mathbb{Q}$, which one calls the field of p-adic numbers, denoted by $\mathbb{Q}_p$.

The algebraic and analytic constructions of $\mathbb{Q}_p$ result in isomorphic fields (see [22]), hence no confusion should arise when talking about *the* field of p-adic numbers. No matter which construction of $\mathbb{Q}_p$ one chooses to work with, there is always the following unique representation of p-adic numbers:

$$\mathbb{Q}_p = \{ \sum_{i=m}^{\infty} a_i p^i \mid a_i \in \{0, 1, 2, \ldots, p-1\}, m \in \mathbb{Z}\}. \tag{1.1}$$

In fact, there is a third construction of the p-adic numbers, using Witt vectors, where (roughly) the above representation is *taken as the definition* of $\mathbb{Q}_p$ (this is quite vague, more on Witt vectors can be found in [43]). We will from now on always use the representation (1.1) when working with $\mathbb{Q}_p$.

Using this representation (1.1), one defines the p-adic integers as

$$\mathbb{Z}_p = \{ \sum_{i=0}^{\infty} a_i p^i \mid a_i \in \{0, 1, 2, ..., p-1\}\},$$

so $\mathbb{Z}_p$ consists of all p-adic numbers with only non-negative powers of p in their p-adic representation.

1.1.2 Basic properties of *p*-adic numbers

There is a map $\mathrm{ord}_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{+\infty\}$, defined by $\mathrm{ord}_p(\sum_{i=m}^{\infty} a_i p^i) = m$ (we assume that $a_m \neq 0$) and $\mathrm{ord}_p(0) = +\infty$, called the *p-adic valuation*. It is immediately clear that ord_p satisfies the following two defining properties of a valuation:

1. $\mathrm{ord}_p(xy) = \mathrm{ord}_p(x) + \mathrm{ord}_p(y)$ for all $x, y \in \mathbb{Q}_p$;
2. $\mathrm{ord}_p(x + y) \geq \min\{\mathrm{ord}_p(x), \mathrm{ord}_p(y)\}$ for all $x, y \in \mathbb{Q}_p$,

where the usual conventions hold regarding to working with $+\infty$.

This valuation induces an absolute value on $\mathbb{Q}_p$ by setting $|x|_p = p^{-\mathrm{ord}_p(x)}$ for nonzero x, and $|0|_p = 0$. Indeed, one easily checks that the three defining properties of an absolute value (see page 7) hold for

$|\cdot|_p$. The absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ actually satisfies a stronger property than the triangle inequality, namely the *strong triangle inequality*: $|x-y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}_p$. Every absolute value satisfying this strong triangle inequality is called *non-Archimedean*, and the corresponding metric space is called a *non-Archimedean metric space*. The fact that the p-adic absolute value is non-Archimedean, makes the topology on $\mathbb{Q}_p$ essentially different from the one on $\mathbb{R}$. There are three immediate consequences of the strong triangle inequality, which we will discuss briefly.

Firstly, the strong triangle inequality implies that there can only be *isosceles triangles* in $\mathbb{Q}_p$, i.e. if x, y and z are p-adic numbers, then $|x-y|_p = |y-z|_p = |z-x|_p$ or, if one of these three absolute values is strictly smaller than an other one, the third one is equal to the largest of the two (see Figure 1.1). This makes it clear, already from the very beginning, that we should be careful using our intuition in a non-Archimedean world, because our intuition is often based on real life experiences, which are in nature Archimedean.

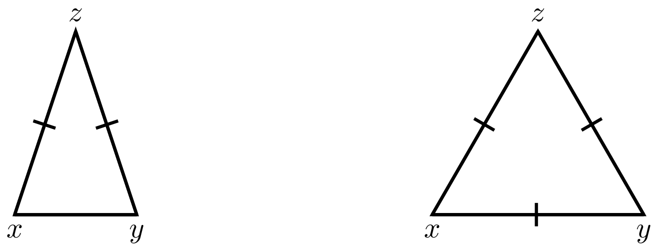


Figure 1.1: Only possible triangles in $\mathbb{Q}_p$

Secondly, something very contra-intuitive happens with the centers of p-adic balls. Let $c \in \mathbb{Q}_p$ and $r \in \mathbb{R}_{\geq 0}$, then the closed ball with center c and radius r is by definition

$$B(c, r) = \{x \in \mathbb{Q}_p \mid |x-c|_p \leq r\}.$$

Open balls are defined analogously, but with a strict inequality. Remark that in $\mathbb{Q}_p$, closed balls are *open and closed* (and open balls are open and closed as well). It follows easily from the strong triangle inequality that every element in $B(c, r)$ is a center of $B(c, r)$ (see [27] for more details on this). Therefore, in a non-Archimedean setting, the words *radius* and *diameter* are synonyms. This makes things even stranger since, opposed to the non existence of non-isosceles triangles, now situations occur that have never been possible in an Archimedean setting. See Figure 1.2 for an illustration of that fact.

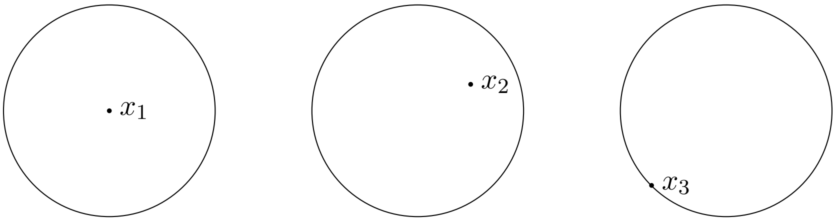


Figure 1.2: Different centers of the same (closed) ball

Thirdly and lastly, there is a severe restriction on the relative position of balls in $\mathbb{Q}_p$. Using the fact that every point in a ball is a center of the ball, one can easily see that two balls should be either contained in one another, or they should be disjoint. However, there is more: if two balls are disjoint, they should be as far away from each other as the radius (or diameter) of the largest ball. It is as if each ball has a territory in which no other points are allowed, see Figure 1.3. Due to practical reasons, however, it will not be possible to always respect this rule when making pictures and illustrations.

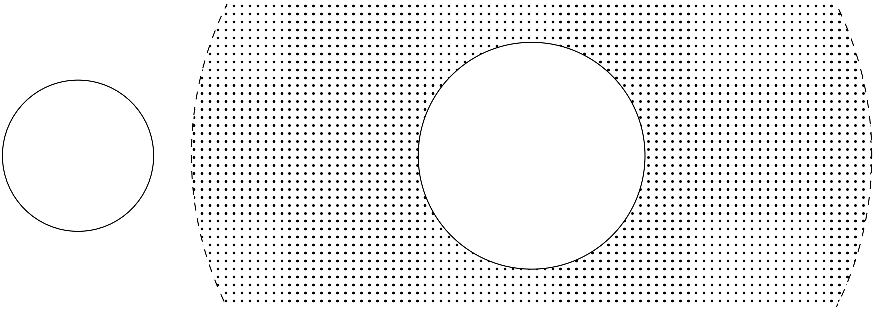


Figure 1.3: Territory of a p-adic ball

It's interesting to note that $\mathbb{Z}_p$ is the closed ball with center 0 and radius 1. Moreover, $\mathbb{Z}_p$ can be used to describe any other closed ball:

$B(c, p^{-l}) = c + p^l\mathbb{Z}_p$ (actually, an arbitrary closed ball in $\mathbb{Q}_p$ is of the form $B(c, r)$, with r a non-negative real number, but since $\mathbb{Q}_p$ is discretely valued, one can always take $r = p^{-l}$, for some $l \in \mathbb{Z}$). In this way we can cover $\mathbb{Z}_p$ with p rescaled copies of itself. If we continue this process of covering, we get a fractal-like picture, illustrated in Figure 1.4, for $p = 3$. Watch out, this picture is *taken* after the third step in this process; in reality, this goes infinitely *deep*, somewhat like a Cantor set. This already gives some evidence that $\mathbb{Q}_p$ is totally disconnected, i.e. the only connected components of $\mathbb{Q}_p$ are the singletons.

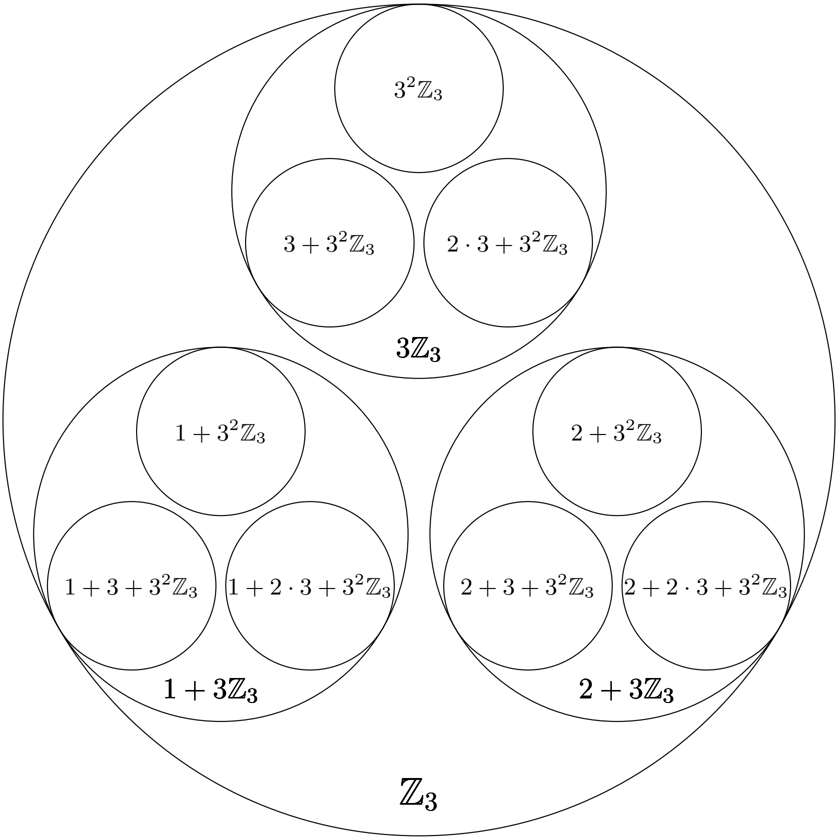


Figure 1.4: A visualisation of $\mathbb{Z}_3$

Next, we introduce the *angular component* of a p-adic number, according to [8]. Before we give the definition, we make an analogy with the complex numbers to justify this name. Let $x+iy$ be a complex number. We can represent this number using polar coordinates: $x+iy = r(\cos\varphi + i\sin\varphi)$, where $|\cos\varphi + i\sin\varphi| = 1$. Since $\cos\varphi + i\sin\varphi$ only depends on φ, it is reasonable to call it the *angular component* of $x+iy$. Remark that r entirely determines the magnitude of $x+iy$. Now let's go back to $\mathbb{Q}_p$: for $x \in \mathbb{Q}_p^\times$ we can always write $x = p^{\mathrm{ord}_p(x)}u$, with $|u|_p = 1$. One calls u the *angular component* of x, denoted by $ac(x) = u$. Remark that also here, obviously, the number $p^{\mathrm{ord}_p(x)}$ entirely determines the magnitude of x, as was the case for its complex counterpart r.

It is, however, more common to call the map

$$\overline{ac} : \mathbb{Q}_p \to \mathbb{Z}_p/p\mathbb{Z}_p : x \mapsto \begin{cases} xp^{-\mathrm{ord}_p(x)} \mod p & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

the *angular component* map. For every nonzero $x \in \mathbb{Q}_p$, this map selects the first nonzero, and therefore the most significant, p-adic digit of x. More generally, one calls the map

$$\overline{\mathrm{ac}}_m : \mathbb{Q}_p \to \mathbb{Z}_p/p^m\mathbb{Z}_p : x \mapsto \begin{cases} xp^{-\mathrm{ord}_p(x)} \mod p^m & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

the *angular component map of depth m*. The depth m acts as a parameter, which can be used to choose the amount of precision to represent a p-adic number. Note that all of these angular component maps are multiplicative.

1.1.3 Some interesting subsets of $\mathbb{Q}_p$

The angular component map plays a significant role in the study of p-adic geometry. To illustrate this, we go back to the example of $\mathbb{Z}_3$. If we look at all the elements of $\mathbb{Z}_3$ with angular component (of depth 1) equal to 1 mod 3, we get a collection of balls

$$\{1+3\mathbb{Z}_3, 3+3^2\mathbb{Z}_3, 3^2+3^3\mathbb{Z}_3, \dots\},$$

see Figure 1.5. Of course one can write

$$\mathbb{Z}_3 = \{x \in \mathbb{Z}_3 \mid \overline{\mathrm{ac}}_1(x) = 1 \mod 3\} \cup \{x \in \mathbb{Z}_3 \mid \overline{\mathrm{ac}}_1(x) = 2 \mod 3\} \cup \{0\},$$

and the same is true for $\mathbb{Z}_3$ replaced by $\mathbb{Q}_3$. This gives a decomposition of $\mathbb{Q}_3$ in sets of a very specific form, which will later turn out to be examples of *p-adic cells* (in dimension one). This is a first, rather trivial, example of what is called a *p-adic cell decomposition* (in dimension one).

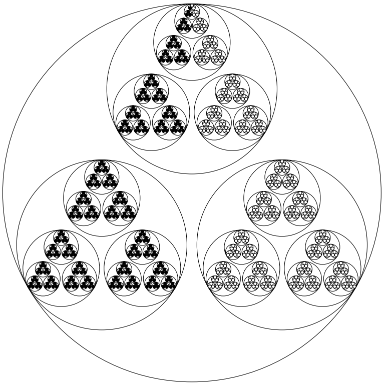


Figure 1.5: Elements with angular component equal to 1 mod 3

More generally, for $\xi \in \mathbb{Q}_p$ the set $\{x \in \mathbb{Q}_p \mid \overline{\mathrm{ac}}_m(x) = \overline{\mathrm{ac}}_m(\xi)\}$ is an (infinite) collection of balls if ξ is nonzero, and is equal to the singleton $\{0\}$ if $\xi = 0$. One can select *one ball* in this collection by putting an extra condition on the order. Indeed, it is easy to calculate that for any nonzero $\xi \in K^\times$ it holds that

$$\{x \in \mathbb{Q}_p \mid \overline{\mathrm{ac}}_m(x) = \overline{\mathrm{ac}}_m(\xi), \mathrm{ord}_p(x) = n\} = B(\xi p^n, p^{-(n+m)}), \qquad (1.2)$$

for every $n \in \mathbb{Z}$.

There are two ways of looking at the set $\{x \in \mathbb{Q}_p \mid \overline{\mathrm{ac}}_m(x) = \overline{\mathrm{ac}}_m(\xi), \mathrm{ord}_p(x) = n\}$, which we will illustrate on the running example $\mathbb{Z}_3$, see Figure 1.6.

A first way is to add horizontal lines to the picture of $\mathbb{Z}_3$, indicating the order. This divides $\mathbb{Z}_3$ into horizontal *strips*, and each of this strips contains a collection of balls. In each strip, there are exactly 2 balls containing elements of a given (nonzero) angular component of depth 1, there are $2 \cdot 3 = 6$ balls containing elements of a given (nonzero) angular component of depth 2, there are $2 \cdot 3^2 = 18$ balls containing elements of

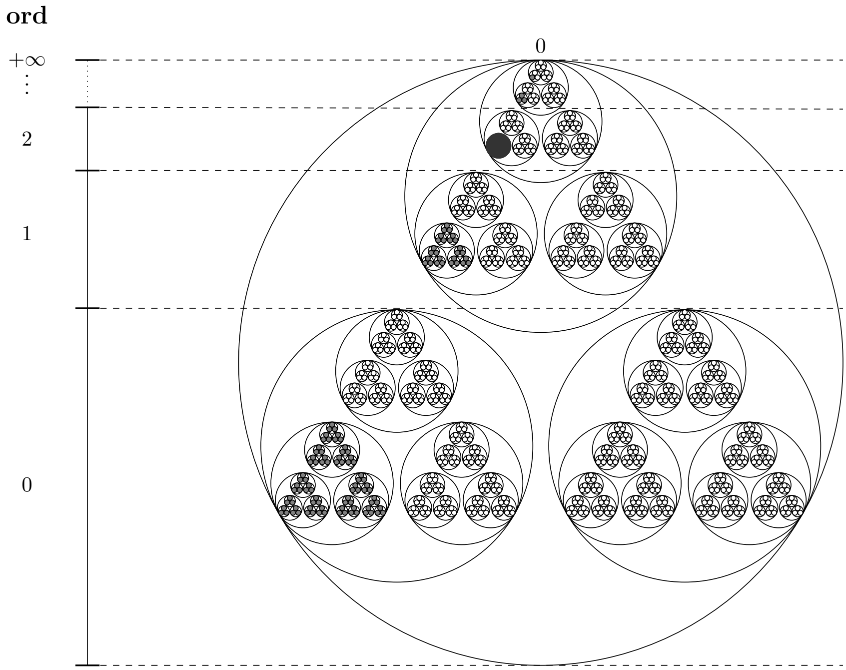


Figure 1.6: The ball $\{x \in \mathbb{Z}_3 \mid \overline{ac}_2(x) = 1+3 \mod 3^2, \mathrm{ord}_p(x) = 2\}$

a given (nonzero) angular component of depth 3, etc. In this way we can for example view $\{x \in \mathbb{Z}_3 \mid \overline{ac}_2(x) = 1+3 \mod 3^2, \mathrm{ord}_3(x) = 2\}$ as selecting from the strip of order 2, one out of six balls. A second way of looking at this set, is first to take all the balls in the collection $\{x \in \mathbb{Z}_3 \mid \overline{ac}_2(x) = 1+3 \mod 3^2\}$, and then to select from this collection the ball in the strip containing elements of order 2.

We saw that every set of the form $\{x \in \mathbb{Q}_p \mid \overline{\mathrm{ac}}_m(x) = \overline{\mathrm{ac}}_m(\xi), \mathrm{ord}_p(x) = n\}$ is a ball, and one could therefore ask the question whether every ball in $\mathbb{Q}_p$ is of this form. Of course this is not the case, since every ball containing 0 contains elements of arbitrarily high order. This is in fact the only objection: every ball $B(c, p^{-l})$ not containing 0, is of the form

$$\{x \in \mathbb{Q}_p \mid \overline{\mathrm{ac}}_m(x) = \overline{\mathrm{ac}}_m(\xi), \mathrm{ord}_p(x) = n\},$$

for some ξ, m and n. To see this, first observe that if $l \leq \mathrm{ord}_p(c)$, then

$0 \in B(c, p^{-l})$. Therefore, if $0 \notin B(c, p^{-l})$, we can write $l = \mathrm{ord}_p(c) + m$, with $m \geq 1$. If we write $c = p^{\mathrm{ord}_p(c)}\xi$, we get:

$$\begin{aligned} B(c, p^{-l}) &= B\big(p^{\mathrm{ord}_p(c)}\xi, p^{-(\mathrm{ord}_p(c)+m)}\big) \\ &= \{x \in \mathbb{Q}_p \mid \overline{\mathrm{ac}}_m(x) = \overline{\mathrm{ac}}_m(\xi), \mathrm{ord}_p(x) = \mathrm{ord}_p(c)\}, \end{aligned}$$

where the last equality follows from (1.2).

If $B(c, p^{-l})$ is a ball that contains 0, we know from the non-Archimedean property that $B(c, p^{-l}) = B(0, p^{-l})$. Let c' be any p-adic number outside $B(0, p^{-l})$, i.e. $\mathrm{ord}_p(c') < l$. Then $B(c', p^{-l})$ doesn't contain 0, so we can write $B(c', p^{-l}) = \{x \in \mathbb{Q}_p \mid \overline{\mathrm{ac}}_m(x) = \overline{\mathrm{ac}}_m(\xi), \mathrm{ord}_p(x) = n\}$, for some ξ and n. But then

$$\begin{aligned} B(0, p^{-l}) &= \{x \in \mathbb{Q}_p \mid x + c' \in B(c', p^{-l})\} \\ &= \{x \in \mathbb{Q}_p \mid \overline{\mathrm{ac}}_m(x + c') = \overline{\mathrm{ac}}_m(\xi), \mathrm{ord}_p(x + c') = n\}. \end{aligned}$$

Combining this with what we saw above, and adjusting notation slightly for later convenience, we find that *every* ball B in $\mathbb{Q}_p$ is of the form

$$B = \{x \in \mathbb{Q}_p \mid \overline{\mathrm{ac}}_m(x - c) = \overline{\mathrm{ac}}_m(\xi), \mathrm{ord}_p(x - c) = n\},$$

where $c \in \mathbb{Q}_p$ and $\xi \in \mathbb{Q}_p^\times$.

Note that the above description of a ball is *not unique*. Later, in a definable context, we will fix m and n, choose ξ out of a (fixed) finite collection of coset representatives and let c vary in a definable way to obtain a definable family of balls.

1.2 *p*-Adic fields

In this section we consider finite field extensions of $\mathbb{Q}_p$. Let K be such a finite field extension of $\mathbb{Q}_p$, then one calls K a *p-adic field*. The aim of this section is to define an absolute value on K, in such a way that the observations from section 1.1 remain valid for K. We give an overview of the properties of K that will be used later in this thesis. The main references for this section are [27] and [22].

1.2.1 Extending $|\cdot|_p$ to p-adic fields

Definition 1. *Let K be a field and let $|\cdot|_1$ and $|\cdot|_2$ be two absolute values on K. Say that $|\cdot|_1$ and $|\cdot|_2$ are* equivalent *if there exists $\alpha \in \mathbb{R}$ such that for all $x \in K$ it holds that $|x|_1 = |x|_2^\alpha$.*

Any absolute value on K induces a metric topology on K, via the obvious metric $d(x,y) = |x-y|$. It can be shown that two equivalent absolute values $|\cdot|_1$ and $|\cdot|_2$ induce the same topology on K, i.e. open sets with respect to $|\cdot|_1$ are also open with respect to $|\cdot|_2$, and vice versa (see [22]).

Let K be a p-adic field. One can view K as a (finite dimensional) $\mathbb{Q}_p$-vector space, and one calls $n = [K : \mathbb{Q}_p] = \dim_{\mathbb{Q}_p} K$ the *degree* of K over $\mathbb{Q}_p$. One says that an absolute value $|\cdot|'$ on K extends an absolute value $|\cdot|$ on $\mathbb{Q}_p$ if $|x|' = |x|$ for all $x \in \mathbb{Q}_p$.

From now on, fix a finite field extension K of $\mathbb{Q}_p$ of degree n. Let $|\cdot|_p$ be the p-adic absolute value on $\mathbb{Q}_p$, as defined in section 1.1. It is not at all obvious that this absolute value extends to an absolute value on K. It is, however, easy to see that if there is an absolute value on K extending $|\cdot|_p$, then it should be unique. For this, one argues that two absolute values $|\cdot|_1$ and $|\cdot|_2$ on K extending $|\cdot|_p$, are automatically equivalent. So there exists $\alpha \in \mathbb{R}$ such that $|x|_1 = |x|_2^\alpha$ for all $x \in K$. Taking for x any nonzero element in $\mathbb{Q}_p$ then gives $\alpha = 1$ (see [22] for more details). Let us now briefly sketch the construction of an absolute value on K extending $|\cdot|_p$.

First, we introduce the *norm* of an element of K over $\mathbb{Q}_p$, following [27].

Definition 2. *Let α be an algebraic element over $\mathbb{Q}_p$ such that $K = \mathbb{Q}_p(\alpha)$, and let $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Q}_p[x]$ be the irreducible polynomial of α over $\mathbb{Q}_p$. Then the* norm of α from K to $\mathbb{Q}_p$ *is defined as either the determinant of the matrix corresponding to the $\mathbb{Q}_p$-linear map "multiplication by α", or equivalently as $(-1)^n a_0$. One denotes the norm of α from K to $\mathbb{Q}_p$ as $\mathbf{N}_{K/\mathbb{Q}_p}(\alpha)$.*

For arbitrary $\beta \in K$ one defines $\mathbf{N}_{K/\mathbb{Q}_p}(\beta)$ as either the determinant of the matrix of "multiplication by β", or equivalently as $\mathbf{N}_{\mathbb{Q}_p(\beta)/\mathbb{Q}_p}(\beta)^{[K:\mathbb{Q}_p(\beta)]}$.

For more details (for example why these definitions are in fact equivalent), we refer to [27] or [22]. Using the norm $\mathbf{N}_{K/\mathbb{Q}_p}$, one finds an absolute value on K extending $|\cdot|_p$ as follows: if K is a degree n extension of $\mathbb{Q}_p$, then $|\alpha|_p = |\mathbf{N}_{K/\mathbb{Q}_p}(\alpha)|_p^{1/n}$ is a non-Archimedean absolute value on K extending the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$, where $|\cdot|_p$ in the right hand side of the definition is the usual p-adic absolute value on $\mathbb{Q}_p$. For the proof, we refer to [27] or [22]. One calls this the *p-adic absolute value on K*.

Using the norm $\mathbf{N}_{K/\mathbb{Q}_p}$ one can also define a *valuation* on K by setting $\mathrm{ord}_p(x) = \mathrm{ord}_p(\mathbf{N}_{K/\mathbb{Q}_p}(x))/n$, for nonzero $x \in K$, and $\mathrm{ord}_p(0) = +\infty$, where ord_p in the right hand side of this definition is the p-adic valuation on $\mathbb{Q}_p$. In other words: $\mathrm{ord}_p(x)$ is the unique rational number satisfying $|x|_p = p^{-\mathrm{ord}_p(x)}$, for nonzero $x \in K$ (see [22] for more on this). One calls ord_p the p-adic valuation on K.

One can easily show that the image $\mathrm{ord}_p(K^\times)$ equals $\frac{1}{e}\mathbb{Z}$, where e is a divisor of n (see [22]). The image $\mathrm{ord}_p(K^\times)$ is called the *value group* of K, denoted by Γ_K. One calls e the *ramification index* of K over $\mathbb{Q}_p$. If $e = 1$, K is called an *unramified* extension of $\mathbb{Q}_p$, and if $e = n$, K is said to be *totally ramified*.

From now on we fix an element $\pi \in K$ such that $\mathrm{ord}_p(\pi) = 1/e$. The following facts are easy to derive, for more details and proofs we refer to [27] and [22]. If one defines

$$\mathcal{O}_K = \{x \in K \mid \mathrm{ord}_p(x) \geq 0\},$$

then $\mathcal{O}_K$ is a local ring with unique maximal ideal

$$\mathcal{M}_K = \{x \in K \mid \mathrm{ord}_p(x) > 0\},$$

and $\mathcal{M}_K$ is generated by π. Of course, $\mathcal{O}_K$ and $\mathcal{M}_K$ serve as analogues in K of $\mathbb{Z}_p$ and $p\mathbb{Z}_p$ in $\mathbb{Q}_p$, respectively. In fact, one immediately sees that $\mathcal{O}_K \cap \mathbb{Q}_p = \mathbb{Z}_p$ and $\mathcal{M}_K \cap \mathbb{Q}_p = p\mathbb{Z}_p$.

Every $x \in K^\times$ can uniquely be written as $x = \pi^m u$, where $u \in \mathcal{O}_K^\times = \mathcal{O}_K \setminus \mathcal{M}_K$ and $m = e \cdot \mathrm{ord}_p(x)$. Since $\mathcal{M}_K$ is a maximal ideal of $\mathcal{O}_K$, $k = \mathcal{O}_K/\mathcal{M}_K$ is a field, which we call the *residue field* of K. In fact, k is a finite field extension of $\mathbb{F}_p$ (note that $\mathbb{F}_p$, using this new terminology, is the residue field of $\mathbb{Q}_p$). If one denotes with f the degree of k over $\mathbb{F}_p$, with e the ramification index of K over $\mathbb{Q}_p$, and with n the degree of K over $\mathbb{Q}_p$, one has that $n = ef$. Note that $\mathbb{Z}$ is a subgroup of $\Gamma_K = \frac{1}{e}\mathbb{Z}$ of index e, so we get the following nice visualization of this formula, see Figure 1.7.

$$\begin{tikzcd}
p\text{-adic field} & \text{residue field} & \text{value group} \\
K \arrow[d, dash, "n"] & k \arrow[d, dash, "f"] & \Gamma_K \arrow[d, dash, "e"] \\
\mathbb{Q}_p & \mathbb{F}_p & \mathbb{Z}
\end{tikzcd}$$

Figure 1.7: The formula $n = ef$

As for p-adic numbers, there also exists a unique p-adic representation of elements of K, as follows. Let A be a fixed (and finite) set of representatives for the cosets of $\mathcal{M}_K$ in $\mathcal{O}_K$, then any $x \in K$ can uniquely be written as

$$x = \sum_{i=m}^{\infty} a_i \pi^i,$$

where $m \in \mathbb{Z}$ and $a_i \in A$ for all $i \geq m$.

1.2.2 Further properties of *p*-adic fields

Although it might seem very convenient that the absolute value $|\cdot|_p$ on K extends the p-adic absolute value on $\mathbb{Q}_p$, and although the formula $n = ef$ is beautiful in nature, there are also reasons to rule against the use of $|\cdot|_p$ on K, depending on the context one works in.

First of all, there is the fact that the value group of K is $\frac{1}{e}\mathbb{Z}$, which is just less convenient to work with than with $\mathbb{Z}$ itself. Since these two groups are isomorphic, one could wonder if there isn't some *rescaling* to have $\mathbb{Z}$ as the value group of K.

Second, one might wonder why it is important to work with an absolute value on K that extends the one on $\mathbb{Q}_p$. In the context we work in, we will actually never need this fact.

This brings us to the third argument: we want to work in the generality of finite field extensions K of $\mathbb{Q}_p$, without constantly having to take into account the ramification index of K over $\mathbb{Q}_p$, when, for example, defining the angular component map on K, or when determining the order of an element, using its p-adic representation.

It is immediate that, in order to have $\mathbb{Z}$ as value group of K, one should define a valuation ord on K as $\operatorname{ord}(x) = e \cdot \operatorname{ord}_p(x)$ for nonzero x, and $\operatorname{ord}(0) = +\infty$, where e is the ramification index of K over $\mathbb{Q}_p$. It is easy to verify that this is indeed a valuation on K. One then defines an absolute value on K as $|x| = q^{-\operatorname{ord}(x)}$ for nonzero x and $|0| = 0$, where q is the cardinality of the residue field k. One then finds that

$$|x| = q^{-\operatorname{ord}(x)} = p^{-fe\cdot\operatorname{ord}_p(x)} = \left(p^{-\operatorname{ord}_p(x)}\right)^n = |x|_p^n,$$

so that $|\cdot|$ and $|\cdot|_p$ are equivalent absolute values. Since equivalent absolute values induce the same topology, we are free to use $|\cdot|$ as an absolute value on K, without having to worry about any topological aspect changing.

So from now on we will work with the valuation ord and the absolute value $|\cdot|$ on K. In this way we can really work with K as we did with $\mathbb{Q}_p$. For example, the order of a nonzero element $x = \sum_{i=m}^{\infty} a_i \pi^i$ will be m, if $a_m \neq 0$, like we are used to from working in $\mathbb{Q}_p$.

There is also an angular component map (of depth m) on K, which is defined in the exact same way as for $\mathbb{Q}_p$:

$$\overline{\mathrm{ac}}_m : K \to \mathcal{O}_K/\pi^m\mathcal{O}_K : x \mapsto \begin{cases} x\pi^{-\operatorname{ord}(x)} \mod \pi^m & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

At this stage, it is good to note that everything from section 1.1 is still valid when $\mathbb{Q}_p$ is replaced by K. By this we mean that the same restrictions on triangles, center of balls and relative position of balls also hold in K. Also the description of balls in terms of the order and the angular component is still valid in K.

The following theorem turns out to be very useful when working in a p-adic context, and will be used intensively throughout this thesis. See [27] for more.

Theorem 3 (Hensel's Lemma)**.** *Let K be a finite field extension of $\mathbb{Q}_p$, and let π be an element with minimal positive order. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{O}_K[x]$ be a polynomial with coefficients in $\mathcal{O}_K$. Suppose there exists $\alpha_0 \in \mathcal{O}_K$ such that $f(\alpha_0) \equiv 0 \mod \pi^{2m+1}$, $f'(\alpha_0) \equiv 0 \mod \pi^m$, but $f'(\alpha_0) \not\equiv 0 \mod \pi^{m+1}$, where f' is the (formal) derivative of f. Then there exists a unique $\alpha \in \mathcal{O}_K$ such that $f(\alpha) = 0$ and $\alpha \equiv \alpha_0 \mod \pi^{m+1}$.*

Note that Hensel's Lemma should be seen as a *lifting theorem*, lifting solutions of $f(x) \equiv 0 \mod \pi^{2m+1}$ to solutions of $f(x) = 0$. Also note that this is a generalised version of the more classically known version of Hensel's Lemma, where one takes $m = 0$ and one drops the condition $f'(\alpha_0) \equiv 0 \mod \pi^m$.

To illustrate the use of Hensel's Lemma, we give some results that are typical for the context of this thesis. From now on we denote by P_n the set of n-th powers in K and by $P_n^\times$ the set of *nonzero* n-th powers in K.

Corollary 4. *Let K be a p-adic field and let $n > 0$ be a positive integer, then there exists a ball $B \subset K$ with center 1, such that $B \subset P_n^\times$.*

Proof. Let $m = \operatorname{ord}(n)$, and let $a \in B(1, q^{-(2m+1)})$. We claim that a is an n-th power. For this to be true, we need to find a solution of $f(x) = 0$ in K, where $f(x) = x^n - a$. Now $f(1) = 1 - a \equiv 0 \mod \pi^{2m+1}$, since $a \in B(1, q^{-(2m+1)})$ and therefore $|1 - a| \leq q^{-(2m+1)}$, which is equivalent with $\operatorname{ord}(1 - a) \geq 2m + 1$. On the other hand, $f'(1) = n \equiv 0 \mod \pi^m$ and $f'(1) = n \not\equiv 0 \mod \pi^{m+1}$, by the choice of $m = \operatorname{ord}(n)$. By Hensel's Lemma, we find a solution in K of $f(x) = 0$, so a is an n-th power. Therefore, every element in $B(1, q^{-(2m+1)})$ is an n-th power, and clearly $0 \notin B(1, q^{-(2m+1)})$, which proves the corollary. □

Corollary 5. *Let K be a p-adic field and let $n > 0$ be a positive integer, then $P_n^\times$ is an open subset of K.*

Proof. Let m be such that $1+\pi^m\mathcal{O}_K \subset P_n^\times$, which we know to exist from the previous corollary (watch out, this is not the same m as in the previous corollary). Then for any $x \in P_n^\times$ we have $x + x\pi^m\mathcal{O}_K \subset P_n^\times$. So there is a ball with center x contained in $P_n^\times$, and since all balls are open in K, we are done. $\square$

Corollary 6. *Let K be a p-adic field, then $P_n^\times$ is a multiplicative subgroup of $K^\times$ of finite index.*

Proof. It is obvious that $P_n^\times$ is a multiplicative subgroup of $K^\times$. By Corollary 4 there exists m such that $1+\pi^m\mathcal{O}_K \subset P_n^\times$. Then clearly there exist a finite number (depending on m) of $\lambda_i \in \mathcal{O}_K^\times$, such that

$$K^\times = P_n^\times \cup \lambda_1 P_n^\times \cup \lambda_2 P_n^\times \cup \cdots \cup \lambda_s P_n^\times,$$

where the unions are disjoint. $\square$

Chapter 2

Introduction to model theory

In this chapter we give an introduction to model theory. Model theory knows two faces, a *pure* one and an *applied* one. This thesis lies on the applied side of model theory, with strong emphasis on (p-adic) geometry. Using tools related to p-adic cell decomposition, we follow in the footsteps of Denef [15], Cluckers-Compte-Loeser [8], and Cluckers-Halupczok [9]. Working with P-minimal structures, we follow ideas of van den Dries [19] and Haskell-Macpherson [23].

In the first section we give a short introduction to first order logic and model theory. In the second section we discuss the main techniques that will be used throughout this thesis.

2.1 Basics from logic and model theory

In this section we introduce some basic notions from logic and model theory. It is not our attempt to be precise, nor to be thorough. This because every reader of this thesis is presumed to be familiar with this material, and if he is not, there are plenty of very decent references on this subject, such as [33] or [5]. The goal of this section is merely to fix notations and to highlight some of the techniques from logic and model theory that are used throughout this thesis. Our main references for this section are the first two chapters of [33].

2.1.1 Languages and structures

The most basic concept in first order logic is that of a *language*. A language determines, as the name suggests, in what words one can talk about a mathematical object. Adding more words to the language, makes the language richer, and more powerful statements can be formulated. More formally: a language $\mathcal{L}$ is a collection of function symbols, relation symbols and constant symbols, denoted by $\mathcal{L} = (\{f_i\}_i, \{R_j\}_j, \{c_k\}_k)$. Interesting examples of languages are:

1. $\mathcal{L}_{\text{ring}} = (+, -, \cdot, 0, 1)$, the ring language;
2. $\mathcal{L}_{\text{Mac}} = (+, -, \cdot, \{P_n\}_{n>0}, 0, 1)$, the Macintyre or semi-algebraic language;
3. $\mathcal{L}_{\text{an}} = \mathcal{L}_{\text{Mac}} \cup ({}^{-1}, \cup_{m>0} K\{x_1, \ldots, x_m\})$, the subanalytic language;
4. $\mathcal{L}_{\text{Pres}} = (+, <, \{\equiv_n\}_{n>0}, 0, 1)$, the Presburger language.

A language is nothing without a *structure* in which the function, relation and constant symbols are actually interpreted as functions, relations and constants. An $\mathcal{L}$-structure $\mathcal{M}$ is a set M, equipped with functions, relations and constants (i.e. elements of M) for all the function, relation and constant symbols that occur in $\mathcal{L}$. We are deliberately vague, since we don't want to go deeper into the formal definitions (for more details, see [33]). One often denotes the structure $\mathcal{M}$ by $(M, \mathcal{L})$ and one calls M the *universe* of the structure $\mathcal{M}$. We give some examples of structures, one for every language in the above list.

1. Let k be any field, then $(k, \mathcal{L}_{\text{ring}})$ is a structure where the addition, subtraction and multiplication on k are the obvious interpretations of $+$, $-$ and $\cdot$, and where the additive and multiplicative neutral elements are interpretations of 0 and 1, respectively. Note that one could as well call $\mathcal{L}_{\text{ring}}$ the *field language*, since one uses the same vocabulary to talk about rings as one does to talk about fields.
2. Let K be a p-adic field, then $(K, \mathcal{L}_{\text{Mac}})$ is a structure where $+$, $-$, $\cdot$, 0 and 1 are interpreted as explained above, and in addition, for every $n > 0$, the relation symbol P_n is interpreted as the subset of

K containing all the n-th powers. This structure is often referred to as the *p-adic semi-algebraic structure.*

3. Let K be a p-adic field, then $(K, \mathcal{L}_{\text{an}})$ is a structure where the symbols from $\mathcal{L}_{\text{Mac}}$ are interpreted as above, where $^{-1}$ is interpreted as the multiplicative inverse extended by $0^{-1} = 0$, and where every function symbol from $K\{x_1, \ldots, x_m\}$ is interpreted as the restricted analytic function $K^m \to K$ given by

$$x \mapsto \begin{cases} f(x) & \text{if } x \in \mathcal{O}_K^m, \\ 0 & \text{otherwise,} \end{cases}$$

where f is a formal power series converging on $\mathcal{O}_K^m$. The structure $(K, \mathcal{L}_{\text{an}})$ is often referred to as the *p-adic subanalytic structure.*

4. Let $\mathbb{Z}$ be the ring of integers, then $(\mathbb{Z}, \mathcal{L}_{\text{Pres}})$ is a structure with the obvious interpretations for $+$, 0 and 1, with the order relation on $\mathbb{Z}$ as an interpretation of $<$, and, for every $n > 0$, the relation $x \equiv y \mod n$ on $\mathbb{Z}$ as an interpretation of $\equiv_n$.

To talk about a given structure $(M, \mathcal{L})$, one uses *formulas.* A formula is a finite string of the following symbols: the symbols coming from $\mathcal{L}$; variable symbols $x, y, z, \ldots, v_1, v_2, \ldots$, or even other symbols; parentheses (and); the equality symbol $=$; the logical connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\to$ (implies) and $\leftrightarrow$ (if and only if); and the quantifiers $\exists$ (there exists) and $\forall$ (for all). Formulas should be syntactically correct, meaning that, for example, parentheses should be balanced. Again, we keep from being precise and rely on the expertise of the reader, or his common sense, regarding these basic notions.

There are two kinds of variables that occur in a formula: the ones that are *bounded* by a quantifier, and the ones that are not. For example, in the $\mathcal{L}_{\text{ring}}$-formula $\varphi = (\exists y)(x \cdot y = 1)$, the variable y is bounded by the existential quantifier, but x is not bounded by any quantifier. The variables that are not bounded by any quantifier are called *free variables.* To emphasize that x occurs freely in the formula φ, one writes $\varphi(x)$. Of course, more than one variable can be free, in which case one uses the notation $\psi(x_1, x_2, \ldots, x_n)$. If there are no free variables in a formula, one calls it a *sentence.*

If one reads the symbols in a formula in the obvious way (as described above), one could wonder whether a given formula in *true* in the universe of the structure one works in. It is, however, immediately clear that, in order for this to be possible, the formula should in fact be a sentence, i.e. containing no free variables. For example if F is a field, how could one possibly answer the question whether $\varphi(x)$, as defined above, is true or not in the structure $(F, \mathcal{L}_{\text{ring}})$? The formula claims that "there exists an element y in F, such that x times y equals 1", but as long as we don't know what x is, we cannot say whether this claim is true or false. The sentence $(\exists x)\varphi(x)$, however, can easily be verified to be true in every field F, since one can take for x any nonzero element in $F^\times$, and for y its multiplicative inverse in F; the sentence $(\forall x)\varphi(x)$ is false in every field, since 0 doesn't have a multiplicative inverse.

It should be mentioned that the concept of *truth* takes some effort to be defined in a precise way: one defines it using induction on the complexity of formulas. Having said this, the reader may be assured that nothing fundamental is hidden, and that his intuition about the truth of formulas is most likely to be correct. The only thing we would like to stress, is that the quantifiers $\exists$ and $\forall$ run over the *entire* universe M of the structure $\mathcal{M}$, as we saw in the example above. If a sentence φ is true in $\mathcal{M}$, one uses the notation $\mathcal{M} \models \varphi$.

It's very important to note that, although a language may contain infinitely many symbols, formulas may only contain a finite number of them. This is the reason why sets and functions that are described by formulas (i.e. definable sets and functions, we will come to this shortly) are considered *easier* than arbitrary ones.

Another interesting remark is that the formula $p_n(x) = (\exists y)(y^n = x)$ completely determines the relation P_n from the Macintyre language. By this we mean that if one replaces in an $\mathcal{L}_{\text{Mac}}$-sentence every occurrence of the relation $P_n(x)$ by the formula $p_n(x)$, one gets an $\mathcal{L}_{\text{ring}}$ sentence with the same truth value. In other words, the expressive power of the language $\mathcal{L}_{\text{Mac}}$ is the same as that of $\mathcal{L}_{\text{ring}}$, and it merely contains convenient abbreviations of formulas of a specific form. However, there are other, very important, reasons for adding the relation symbols P_n to the ring language, as we will see shortly.

Let $\mathcal{M} = (M, \mathcal{L})$ be a structure. Then every formula $\varphi(x_1, \ldots, x_n)$ determines a subset of M^n, namely

$$\varphi(M) = \{(a_1, \ldots, a_n) \in M^n \mid \mathcal{M} \models \varphi(a_1, \ldots, a_n)\}.$$

One calls a subset $X \subset M^n$ *$\mathcal{L}$-definable* if there exists an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ such that $X = \varphi(M)$. We say that $\varphi(x_1, \ldots, x_n)$ *defines* X, or conversely, that X is *defined* by $\varphi(x_1, \ldots, x_n)$. A function $f : M^r \to M^s$ is said to be *definable* if its graph is a definable subset of M^{r+s}. The concept of definability plays a central role in the research area we work in. Since definable sets and functions are encoded with only a finite amount of data, they are considered to be easier in nature, and more likely to possess nice properties; we go deeper into this in subsection 2.2.2. However, one should not be tempted to think that all definable sets and functions are of a nice and simple form, since a lot of quantifiers in the defining formula may give quite a complicated result.

Let $X \subset M^r \times M^s$ be a definable set. Say that a definable function $f : M^r \to M^s$ is a *definable selection* function for X, if for each $(t_1, t_2) \in X$ with $t_1 \in M^r$ and $t_2 \in M^s$, the point $(t_1, f(t_1))$ lies in X.

Let $\mathcal{M} = (M, \mathcal{L})$ be a structure and let $X, Y \subset M^n$ be definable sets, defined by φ and ψ, respectively. Then $\varphi \wedge \psi$ defines $X \cap Y$, $\varphi \vee \psi$ defines $X \cup Y$ and $\neg\varphi$ defines $M^n \setminus X$. Moreover, $(\exists y)\varphi(x, y)$, where $x = (x_1, \ldots, x_{n-1})$, defines the projection of X onto the first $n-1$ coordinates of X in M^{n-1}. See Figure 2.1 for a visualization of this last fact.

Let $(M, \mathcal{L})$ be a structure and let $\mathcal{L}'$ be the language $\mathcal{L} \cup (\{c_m\}_{m \in M})$ obtained by adding as much constant symbols to $\mathcal{L}$ as there are elements in M. Let $(M, \mathcal{L}')$ be the structure where every constant symbol c_m is interpreted as the element $m \in M$. Then one says a set $X \subset M^n$ is $\mathcal{L}$-definable *with parameters (from M)* if X is $\mathcal{L}'$-definable. This is just a complicated way of saying that in the formula defining X, one may use constants from the universe. For example: $X = \{a \in \mathbb{Z} \mid a < 2\}$ is $\mathcal{L}_{\text{Pres}}$-definable with parameters. From now on, by “definable” we will always (often implicitly) mean “definable with parameters”.

Let $\mathcal{L}$ be a language. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are said to be *elementary equivalent* if for every $\mathcal{L}$-sentence φ, one has that $\mathcal{M} \models \varphi$ if

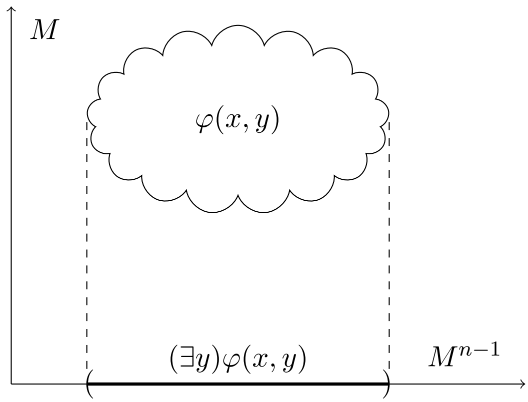


Figure 2.1: Projection using the $\exists$ quantifier

and only if $\mathcal{N} \models \varphi$. If $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent, one writes $\mathcal{M} \equiv \mathcal{N}$, and to incorporate the used language into this notation, one sometimes also writes $\mathcal{M} \equiv_{\mathcal{L}} \mathcal{N}$. Structures being elementary equivalent or not, depends heavily on the language one works with. If $\mathcal{M} \equiv_{\mathcal{L}} \mathcal{N}$, this merely means that $\mathcal{L}$ hasn't got enough expressive power to express the fact that $\mathcal{M}$ and $\mathcal{N}$ are different. In other words, from an $\mathcal{L}$-point of view $\mathcal{M}$ and $\mathcal{N}$ look exactly the same, although they may very well be very different. For example: $\mathbb{C} \equiv_{\mathcal{L}_{\text{ring}}} \mathbb{Q}^{\text{alg. cl.}}$, because the theory of algebraically closed fields of characteristic zero is complete (see section 2.1.2 for the definition of a theory, and [33] for the definition of a complete theory). However, $\mathbb{C}$ and $\mathbb{Q}^{\text{alg. cl.}}$ are not isomorphic as fields, because $\mathbb{C}$ is uncountable and $\mathbb{Q}^{\text{alg. cl.}}$ is countable. As another example of elementary equivalent structures, one has that $\mathbb{Z} \equiv \mathbb{Q} \oplus \mathbb{Z}$ in the Presburger language, where on $\mathbb{Q} \oplus \mathbb{Z}$ the addition is componentwise, the order is interpreted as the lexicographical order, 0 is interpreted as $(0,0)$, 1 as $(0,1)$, and $\equiv_n$ is interpreted as the relation $(r,l) \equiv_n (s,k)$ if and only if $n \mid (r,l) - (s,k)$.

A notion that extends that of a structure, is a *many-sorted structure*. The definition of a many-sorted structure isn't hard, and a detailed exposition can be found in [21]. However, we prefer to give an illustrative example to introduce this notion rather than to go into too much details. The example that we give is that of the *Denef-Pas* language, which was first introduced by Pas in [35], and is often used, also recently in e.g. [13], in the context of motivic integration.

One defines the three-sorted language of Denef-Pas as follows:

$$\mathcal{L}_{\text{Denef-Pas}} = (\mathcal{L}_{\text{ring}}, \mathcal{L}_{\text{ring}}, \mathcal{L}_{\text{Pres}}, \text{ord}, \overline{\text{ac}}),$$

where $\mathcal{L}_{\text{ring}}$ and $\mathcal{L}_{\text{Pres}}$ are the languages as defined above, and ord and $\overline{\text{ac}}$ are function symbols. The Denef-Pas language is designed to talk simultaneously about a p-adic field K, its residue field k and its value group $\mathbb{Z}$. An example of a three-sorted $\mathcal{L}_{\text{Denef-Pas}}$-structure is therefore

$$((K, \mathcal{L}_{\text{ring}}), (k, \mathcal{L}_{\text{ring}}), (\mathbb{Z}, \mathcal{L}_{\text{Pres}}), \text{ord}, \overline{\text{ac}}).$$

Moreover, $\text{ord} : K^{\times} \to \mathbb{Z}$ denotes the p-adic valuation, and $\overline{\text{ac}} : K \to k$ denotes the angular component map. The universes K, k and $\mathbb{Z}$ are called the *sorts* of the structure. See [13] for more details and more precise statements concerning this structure.

An $\mathcal{L}_{\text{Denef-Pas}}$-formula is an ordinary first order formula, using function, relation and constant symbols from $\mathcal{L}_{\text{ring}}$, $\mathcal{L}_{\text{Pres}}$, ord and $\overline{\text{ac}}$, with the only difference being that the quantifiers are now *typified.* This means that there are three existential and three universal quantifiers, each specifying variables from a specific sort K, k or $\mathbb{Z}$. Examples of $\mathcal{L}_{\text{Denef-Pas}}$-formulas are $(\forall x \in K)(\exists y \in K)(\text{ord}(x) = \text{ord}(y))$ and $(\forall \xi \in k)(\exists x \in K)(\overline{\text{ac}}(x) = \xi \wedge \text{ord}(x) \geq 0)$.

2.1.2 Theories and models

Let $\mathcal{L}$ be a language and $\mathcal{M}$ an $\mathcal{L}$-structure. An $\mathcal{L}$-*theory* T is a (possibly infinite) set of $\mathcal{L}$-sentences. One says that $\mathcal{M}$ is a *model* of T if $\mathcal{M} \models \varphi$ for every φ in T, and one writes $\mathcal{M} \models T$. “Model Theory” owes his name to the fact that it studies the different *models* of a given theory, and was first called this way by Tarski in [45]. Let φ be an $\mathcal{L}$-sentence, then one says that φ is a *logical consequence* of T, if $\mathcal{M} \models \varphi$ for every model $\mathcal{M}$ of T. Given a structure $(M, \mathcal{L})$, one calls $\text{Th}(M, \mathcal{L})$ the *full theory* of $(M, \mathcal{L})$, consisting of all $\mathcal{L}$-sentences that are true in M.

A theory T is said to be *satisfiable* if there exists a model $\mathcal{M}$ such that $\mathcal{M} \models T$. A celebrated theorem says that a theory T is satisfiable if and only if every finite subset of T is satisfiable. This is a direct consequence of Gödels *completeness theorem*, and is referred to as the *compactness theorem*.

A theory can be seen as a kind of axiom system, and a model of the theory as a mathematical structure that satisfies all the axioms. For example, the theory of fields of characteristic zero contains the field axioms, together with the sentences $1 \neq 0$, $1+1 \neq 0$, $1+1+1 \neq 0$, etc. Every model of this theory will therefore be a field of characteristic zero. Note that it's not possible to express with only a finite number of sentences the fact that a field has characteristic zero. This can be used to prove theorems for fields of characteristic zero, by proving them first for fields of arbitrary large positive characteristic and then by using compactness. An example of this is the Ax-Grothendieck theorem, stating that every injective polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is automatically surjective (see e.g. [44]).

As we already saw in subsection 2.1.1, a definable set becomes more complicated when there are a lot of quantifiers in the defining formula. One therefore considers a definable set to be *easy* if it can be defined with a quantifier free formula. More generally, one considers a theory T to be *easy* if every definable set in every model of T can be defined with a quantifier free formula. Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. If for every $\mathcal{L}$-formula φ there is a quantifier free $\mathcal{L}$-formula ψ such that $T \models \varphi \leftrightarrow \psi$, one says that T has (or *admits*) *quantifier elimination* for $\mathcal{L}$. Here, one uses the abbreviation $T \models \varphi(x_1, \dots, x_n)$ to mean $T \models (\forall x_1, \dots, x_n)\varphi(x_1, \dots, x_n)$.

Several theories have been proven to admit elimination of quantifiers. We give a brief overview in increasing order of importance for the context that we work in.

1. Let $\mathcal{L}_{\text{ring}}$ be the ring language, and ACF the $\mathcal{L}_{\text{ring}}$-theory of algebraically closed fields, i.e. the theory containing the field axioms and for every $n \geq 1$ the sentence $(\forall a_0, \dots, a_{n-1})(\exists x)(x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0)$. Then ACF admits elimination of quantifiers for $\mathcal{L}_{\text{ring}}$ (see e.g. [33]). Although we won't be working with algebraically closed fields, we include this example because it has a nice geometrical interpretation.

Let K be an algebraically closed field. Call a set $X \subset K^n$ *constructible* if it is a finite Boolean combination of Zariski closed sets. Here, *Zariski closed* means the zero locus of a finite set of polynomials over K. Then $X \subset K^n$ is $\mathcal{L}_{\text{ring}}$-definable if and only if it is constructible (see e.g. [33]). This has two important consequences: firstly, the image of a constructible set under a polynomial map is constructible (this is Chevalley's Theorem), and secondly, if $X \subset K$ is definable, then either X or $K \setminus X$ is finite. This second property is also referred to as *strong minimality*, a concept that we will meet again in the study of P-minimal structures, see chapter 4.

2. In this example we consider real closed fields, a generalization of the field of real numbers. Say that a field F is *real closed* if every odd degree polynomial $f \in F[x]$ has a zero in F, and every element or its negative is a square in F, or equivalently if F is $\mathcal{L}_{\text{ring}}$-elementary equivalent to $\mathbb{R}$ (see [39] for more on real closed fields). Examples of real closed fields are $\mathbb{Q}^{\text{alg. cl.}} \cap \mathbb{R}$, the field of real algebraic numbers, and $\mathbb{R}$ itself. It can be shown that only the algebraically closed fields admit quantifier elimination in the ring language (see [33]), so the theory of real closed fields does not admit quantifier elimination in $\mathcal{L}_{\text{ring}}$. However, this theory *does* have quantifier elimination in $\mathcal{L}_{\text{or}} = \mathcal{L}_{\text{ring}} \cup \{<\}$, the language of ordered rings (here, a real closed field is considered to be an ordered field, where the ordering is respected by addition and multiplication with positive elements; it can be shown that every real closed field is also an ordered field, see [33]).

3. Let $\mathcal{L}_{\text{Pres}}$ be the Presburger language, and $\text{Th}(\mathbb{Z}, \mathcal{L}_{\text{Pres}})$ the full theory of $(\mathbb{Z}, \mathcal{L}_{\text{Pres}})$, as defined above. Then $\text{Th}(\mathbb{Z}, \mathcal{L}_{\text{Pres}})$ admits elimination of quantifiers (see [38]). This, together with other facts about $(\mathbb{Z}, \mathcal{L}_{\text{Pres}})$ (which can be found in [6]), is used in the Preparation Theorem of [9], which is essential for the results in chapter 3.

4. A p-adic analogue of real closed fields, are *p-adically closed fields* (see [40] for the main reference on p-adically closed fields). A p-adically closed field K is a p-valued field that is Henselian and that has as value group a $\mathbb{Z}$-group (i.e. $\mathcal{L}_{\text{Pres}}$-elementary equivalent

to $\mathbb{Z}$). Here a p-valued field is a valued field K of characteristic zero, of which the residue field k has characteristic p and is a finite dimensional vector space over $\mathbb{F}_p$. Equivalently, a field is p-adically closed if and only if it is $\mathcal{L}_{\text{ring}}$-elementary equivalent to a finite field extension of $\mathbb{Q}_p$. Let $\mathcal{L}_{\text{Mac}}$ be the Macintyre language, then the theory of p-adically closed fields admits quantifier elimination for $\mathcal{L}_{\text{Mac}}$. This result is due to Prestel and Roquette (see [40]), where they generalize a proof of Macintyre for quantifier elimination for $\mathbb{Q}_p$ (see [31]). Generalizing the terminology from example 2 on page 24, one also calls $(K, \mathcal{L}_{\text{Mac}})$ a *p-adic semi-algebraic structure* if K is a p-adically closed field.

Some remarks should be made. Firstly, since we (implicitly) allow parameters from K in our language, we don't need to include constant symbols to $\mathcal{L}_{\text{Mac}}$ for a basis of the residue field k over $\mathbb{F}_p$, as was done in [40]. Secondly, in [40] and [31], one adds a unary relation symbol to $\mathcal{L}_{\text{Mac}}$, to indicate the valuation ring. However, this is unnecessary, since a nonzero element x lies in the valuation ring if and only if $1+\pi x^2$ is a square, if char $K \neq 2$, or if $1+\pi x^3$ is a cube, if char $K = 2$ (see the proof of Lemma 7 later on, where this fact is proved). Thirdly, where in [40] and [31] model theory is used to prove quantifier elimination, in [16] Denef gives a completely algebraic proof of quantifier elimination for $\mathbb{Q}_p$ in $\mathcal{L}_{\text{Mac}}$.

5. Let K be a p-adic field. Let $\mathcal{L}_{\text{an}}$ be the subanalytic language and $\text{Th}(K, \mathcal{L}_{\text{an}})$ the full theory of $(K, \mathcal{L}_{\text{an}})$. Then $\text{Th}(K, \mathcal{L}_{\text{an}})$ admits quantifier elimination for $\mathcal{L}_{\text{an}}$, as was proved in [17] and [20].

One sees in the examples above that quantifier elimination often only holds after adding some relation symbols to the language. Typically, this new relation can be defined in the original language, but with the use of quantifiers. In this way, the quantifiers in the formulas are *hidden* by the new relation symbols. One could then say that there is quantifier elimination *up to* quantifiers occurring in formulas of a specific form. In example 2 on page 31, this is the relation $<$, which is $\mathcal{L}_{\text{ring}}$-definable: $x < y$ if and only if $(\exists z)(z \neq 0 \wedge x + z^2 = y)$. As Marker puts it in [33], the ordering is *the only obstruction* to quantifier elimination for real closed fields. In example 3 on page 31, the relations $\equiv_n$ are added to the ordered group language $\mathcal{L}_{\text{group}} = (+, <, 0, 1)$, for all $n > 0$. Again, these relations are $\mathcal{L}_{\text{group}}$-definable, because $x \equiv_n y$ if and only if $(\exists z)(x - y = nz)$. In examples 4 on page 31 and 5 on page 32, the unary relations P_n are added to the ring language for every $n > 0$. Of course P_n is $\mathcal{L}_{\text{ring}}$-definable, since $x \in P_n$ if and only if $(\exists y)(x = y^n)$.

Often, one puts extra conditions on a given structure to make sure that certain nice properties are satisfied by definable sets and functions in this structure. Two main examples of this are:

1. o-Minimal structures (see [37] and [19] for the main references on this subject). Let $\mathcal{L}_{\text{or}}$ be the language of ordered rings and let $\mathcal{R} = (R, \mathcal{L}_{\text{or}})$ be an $\mathcal{L}_{\text{or}}$-structure, where R is a dense linearly ordered set without endpoints. Then $\mathcal{R}$ is said to be an *o-minimal structure* if every $\mathcal{L}_{\text{or}}$-definable subset of R is a finite union of intervals and points. Examples of o-minimal structures are real closed fields (see example 2 on page 31). We give the argument of Marker in [33]: by quantifier elimination of real closed fields in the language $\mathcal{L}_{\text{or}}$, every definable subset of R is a finite Boolean combination of sets of the form $\{x \in R \mid p(x) = 0\}$ and $\{x \in R \mid q(x) < 0\}$, where p and q are polynomials with coefficients in R. A set of the first form is either R or a finite number of isolated points; a set of the second form is a finite union of intervals. Therefore, any $\mathcal{L}_{\text{or}}$-definable set is a finite union of intervals and points. Pillay and Steinhorn proved in [37] that the converse is also true: any o-minimal ordered ring is a real closed field.

   Another example of an o-minimal structure is $(\mathbb{R}, \mathcal{L}_{\text{exp}})$, where $\mathcal{L}_{\text{exp}} = (+, -, \cdot, <, \exp, 0, 1)$, with exp interpreted as $\mathbb{R} \to \mathbb{R} : x \mapsto e^x$ (see [46] for more on this, and [47] for other o-minimal structures).

2. P-minimal structures (see [23] for the main reference on this topic). Let $\mathcal{L}$ be a language extending the ring language $\mathcal{L}_{\text{ring}}$, and let K be a p-valued field whose value group is a $\mathbb{Z}$-group (see example 4 on page 31). The structure $(K, \mathcal{L})$ is said to be *P-minimal* if for every $\mathcal{L}$-elementary equivalent structure $(K', \mathcal{L})$, any definable set $X \subset K'$ is quantifier free $\mathcal{L}_{\text{Mac}}$-definable (with parameters from K'). This implies that every definable subset of K is a finite union

of open sets and points. Indeed, quantifier free $\mathcal{L}_{\text{Mac}}$-definable subsets of K are finite Boolean combinations of sets of the form $\{x \in K \mid p(x) = 0\}$ and $\{x \in K \mid q(x) \in P_n^\times\}$, where p and q are polynomials with coefficients in K. A set of the first form is either K or a finite number of isolated points; a set of the second form is open, since $P_n^\times$ is open (see Corollary 5) and the preimage of an open set under a continuous map is open. The complement of a set of the first form is either empty or K minus a finite number of points, which is open; the complement of a set of the second form is the union of the zero locus of $q(x)$ and a finite number of sets of the form $\{x \in K \mid q(x) \in \lambda P_n^\times\}$, which are all open (see Corollary 6). This justifies the name "P-minimality" as an analogue of o-minimality.

As another instance of this analogy, Haskell and Macpherson proved in [23] that any P-minimal expansion of an $\mathcal{L}_{\text{ring}}$-structure is p-adically closed. A first class of examples of P-minimal structures are the p-adic semi-algebraic structures (as defined in example 4 on page 31), as can easily be seen using quantifier elimination. A less trivial class of examples of P-minimal structures are the p-adic subanalytic structures: those were proved to be P-minimal in [20]. There are also intermediate structures in between semi-algebraic and subanalytic structures that are P-minimal (see [10]).

There are also other minimality conditions one can put on structures, such as for example C-minimality (see [32]) and b-minimality (see [12]). Later, in section 2.2.2, we will explore some of the properties of definable sets and functions in o-minimal and P-minimal structures. Moreover, the entire chapter 4 is devoted to new research on P-minimal structures.

2.2 Main techniques

In this section we come to the core business of this thesis: p-adic cell decomposition and tame geometry. p-Adic cell decomposition, together with related cell-decomposition theorems, forms the basic technique that we will use in chapter 3 to study the behavior of definable Lipschitz functions on p-adic fields. In chapter 4 we develop new tameness results for functions in P-minimal structures.

2.2.1 *p*-Adic cell decomposition

Over time, a lot of cell decomposition theorems have been formulated, in a wide variety of contexts. Examples are the following structures, who all admit some kind of cell decomposition:

1. o-minimal structures (see [19]);
2. Presburger structures (see [6]);
3. P-minimal structures endowed with definable selection (see [34]);
4. p-adic semi-algebraic structures (see [14], [15] and [16]);
5. p-adic subanalytic structures (see [7]).

Examples 1 and 2 will not be discussed in this thesis; for more information, we refer to the above references. Example 3 will be treated in chapter 5. Examples 4 and 5 will be of main importance and will be treated simultaneously under the name *p-adic cell decomposition*. Note that over the years, the formulation of cell decomposition theorems has been modified to meet more modern standards. We will therefore use [9] and [8] as our main references for the formulation and the basic results of p-adic cell decomposition.

From now on, and until the end of this chapter, let K denote a p-adic field and let $\mathcal{L}$ be either the semi-algebraic or the subanalytic language. So $\mathcal{L}$-*definable*, or just *definable*, will from now on mean either $\mathcal{L}_{\text{Mac}}$-definable or $\mathcal{L}_{\text{an}}$-definable. In this setup, we formulate the concept of p-adic cell decomposition.

Before we give a precise definition of p-adic cells, we go back to chapter 1. In this chapter, we saw that every ball B in K is of the form

$$B = \{x \in K \mid \overline{\text{ac}}_m(x-c) = \overline{\text{ac}}_m(\xi), \text{ord}(x-c) = n\}.$$

The following two lemmas show that every ball is definable in $\mathcal{L}$.

Lemma 7. *Let $n \in \mathbb{Z}$ and $c \in K$. Then $\{x \in K \mid \operatorname{ord}(x-c) = n\}$ is an $\mathcal{L}$-definable subset of K.*

Proof. It suffices to prove that $\{(x,y) \in K^2 \mid \operatorname{ord}(x) \leq \operatorname{ord}(y)\}$ is $\mathcal{L}$-definable, say by the $\mathcal{L}$-formula $\varphi(x,y)$. Indeed, then

$$\{x \in K \mid \operatorname{ord}(x-c) = n\} = \{x \in K \mid K \models \varphi(x-c, \pi^n) \wedge \varphi(\pi^n, x-c)\}.$$

Slightly modifying the approach of [17] to p-adic fields (instead of $\mathbb{Q}_p$), it holds that $\operatorname{ord}(x) \leq \operatorname{ord}(y)$ if and only if $y = 0$ or $x^2 + \pi y^2 \in P_2$, if char $K \neq 2$, or $x^3 + \pi y^3 \in P_3$ if char $K = 2$. We give a proof of this statement in the case that char $K \neq 2$, the other case is similar.

$\boxed{\Rightarrow}$ We may suppose that y, and therefore also x, is nonzero. We have to prove that $x^2 + \pi y^2 \in P_2$. Let us first assume that $x \in \mathcal{O}_K^\times$, so $0 = \operatorname{ord}(x) \leq \operatorname{ord}(y)$. We use Hensel's Lemma: let $F(X) = X^2 - (x^2 + \pi y^2)$. Then $F(x) \equiv 0 \mod \pi$ and $F'(x) = 2x \not\equiv 0 \mod \pi$, so the solution of $F(X) \equiv 0 \mod \pi$ lifts to a solution of $F(X) = 0$, hence $x^2 + \pi y^2$ is a square. If $x \in K^\times$ is arbitrary, and if $\operatorname{ord}(x) \leq \operatorname{ord}(y)$, then $0 = \operatorname{ord}(\pi^{-\operatorname{ord}(x)}x) \leq \operatorname{ord}(\pi^{-\operatorname{ord}(x)}y)$, hence $\pi^{-2\operatorname{ord}(x)}(x^2 + \pi y^2) \in P_2$ by the above argument, and therefore also $(x^2 + \pi y^2) \in P_2$.

$\boxed{\Leftarrow}$ If y were zero, then obviously $\operatorname{ord}(x) \leq \operatorname{ord}(y)$, so we may assume $y \neq 0$. If $\operatorname{ord}(x) > \operatorname{ord}(y)$, then $\operatorname{ord}(x^2 + \pi y^2) = \operatorname{ord}(\pi y^2)$ is odd, while every square has even order, a contradiction.

This proves the statement and therefore the lemma. $\square$

Actually, from the proof of the previous lemma we can already conclude that every ball is definable in K. Still the following lemma is interesting and we will need it later on.

Lemma 8. *Let $c \in K$, $\xi \in K^\times$, and let $m \geq 1$ be an integer. Then $\{x \in K \mid \overline{\mathrm{ac}}_m(x-c) = \overline{\mathrm{ac}}_m(\xi)\}$ is an $\mathcal{L}$-definable set.*

Proof. We will prove the lemma for $m = 1$, the proof for general m is analogous. It suffices to prove that $\{x \in K \mid \overline{\mathrm{ac}}_1(x) = 1\}$ is definable by an $\mathcal{L}$-formula, say $\psi(x)$, since then

$$\{x \in K \mid \overline{\mathrm{ac}}_1(x-c) = \overline{\mathrm{ac}}_1(\xi)\} = \{x \in K \mid K \models \psi((x-c)\xi^{-1})\}.$$

Let $P_{q-1}^{\times}$ denote the nonzero $(q-1)$-th powers in K, where q is the cardinality of the residue field of K. It is standard that $x^{q-1} = 1$ in $\mathbb{F}_q$, the field with q elements (see e.g. [27]). We claim that

$$\{x \in K \mid \overline{\mathrm{ac}}_1(x) = 1\} = P_{q-1}^{\times} \cup \pi P_{q-1}^{\times} \cup \cdots \cup \pi^{q-2} P_{q-1}^{\times}.$$

Once the claim is proved, we are done, since the right hand side of the above formula is clearly $\mathcal{L}$-definable. We prove the claim by showing that two inclusions hold.

$\boxed{\supset}$ Let $x \in \pi^i P_{q-1}^{\times}$. Then we can write $x = \pi^i y^{q-1}$, for some nonzero $y \in K$, hence

$$\begin{aligned} \overline{\mathrm{ac}}_1(x) &= \overline{\mathrm{ac}}_1(y^{q-1}) \\ &= \overline{\mathrm{ac}}_1(y)^{q-1} \\ &= 1. \end{aligned}$$

$\boxed{\subset}$ Let $x \in K$, $\overline{\mathrm{ac}}_1(x) = 1$, and $\mathrm{ord}(x) \equiv i \mod q-1$. We will prove that $x \in \pi^i P_{q-1}^{\times}$. Let $x' = \pi^{-\mathrm{ord}(x)} x \in \mathcal{O}_K^{\times}$, then $\overline{\mathrm{ac}}_1(x') = 1$. Let $F(X) = X^{q-1} - x'$, then $F(1) = 1 - x' \equiv 0 \mod \pi$ and $F'(1) = q - 1 \not\equiv 0 \mod \pi$, because $q \equiv 0 \mod \pi$, and $0 \not\equiv 1 \mod \pi$. Hence we can lift the solution of $F(X) \equiv 0 \mod \pi$ to a solution of $F(X) = 0$ using Hensel's Lemma, so $x' \in P_{q-1}^{\times}$. We then conclude by noting that $x = \pi^{\mathrm{ord}(x)} x' \in \pi^i P_{q-1}^{\times}$. □

The previous two lemmas show that every set that is defined by specifying the angular component of some depth and specifying the order, is definable in $\mathcal{L}$.

A very basic form of p-adic cell decomposition states that every $\mathcal{L}$-definable subset of K can be decomposed in a finite partition consisting of *cells*, which are definable sets of a very specific form, namely sets with certain constraints on the angular component and the order. We have in fact already seen some very basic examples of cells in chapter 1, such as $\{x \in \mathbb{Q}_3 \mid \overline{\mathrm{ac}}_1(x) = 1, \mathrm{ord}(x) \geq 0\}$ (see Figure 1.5) and $\{x \in \mathbb{Q}_3 \mid \overline{\mathrm{ac}}_2(x) = 1 + 3 \mod 3, \mathrm{ord}(x) = 2\}$ (see Figure 1.6). To see which other conditions are natural to expect in formulas defining $\mathcal{L}$-definable sets, we consider the following example in $\mathbb{Q}_3$.

Let $X = \{x \in \mathbb{Q}_3 \mid (\exists y)(y^2 = x)\}$. Of course this is just the set P_2 of squares in $\mathbb{Q}_3$, but we will try to define X using conditions on the angular component and the order only. We claim that

$$X = \{x \in \mathbb{Q}_3 \mid \overline{\mathrm{ac}}_1(x) = 1, \mathrm{ord}(x) \in 2\mathbb{Z}\} \cup \{0\}.$$

We prove the claim by showing two inclusions:

$\boxed{\subset}$ Let $x \in \mathbb{Q}_3^\times$ be a nonzero square, then obviously $\mathrm{ord}(x) \in 2\mathbb{Z}$. Let $y \in \mathbb{Q}_3^\times$ be such that $y^2 = x$. Then $\overline{\mathrm{ac}}_1(x) = \overline{\mathrm{ac}}_1(y^2) = \overline{\mathrm{ac}}_1(y)^2$, so $\overline{\mathrm{ac}}_1(x)$ is a nonzero square in $\mathbb{F}_3$, therefore $\overline{\mathrm{ac}}_1(x) = 1$ (because 2 is not a square in $\mathbb{F}_3$).

$\boxed{\supset}$ This is an easy consequence of Hensel's lemma. Let $x \in \mathbb{Q}_3^\times$ be such that $\overline{\mathrm{ac}}_1(x) = 1$ and $\mathrm{ord}(x) \in 2\mathbb{Z}$, say $\mathrm{ord}(x) = 2l$. Then $x = 3^{2l}(1 + a_1 \cdot 3 + a_2 \cdot 3^2 + \cdots)$, and we write $u = x \cdot 3^{-2l}$. Let $F(X) = X^2 - u$, then $F(1) \equiv 0 \mod 3$ and $F'(1) = 2 \not\equiv 0 \mod 3$, so u is a square by Hensel's lemma. Then also x is a square.

In the above argument we saw that there are two possible objections for a nonzero $x \in \mathbb{Q}_3^\times$ to be a square. Firstly $\overline{\mathrm{ac}}_1(x)$ should be a nonzero square in $\mathbb{F}_3^\times$, and secondly $\mathrm{ord}(x)$ should be even. When using the visual interpretation of $\mathbb{Q}_3$ from chapter 1, we get the picture in Figure 2.2.

Very vaguely one could say that about one in four 3-adic numbers are a square. More formally, one could say that $P_2^\times$ has index four in $\mathbb{Q}_3^\times$. The same is true for all primes $p \neq 2$, namely $\mathbb{Q}_p^\times / P_2^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The same argument holds, since in $\mathbb{F}_p^\times$, half of the elements are squares. For $p = 2$, the situation is somewhat different, namely $P_2^\times$ has index eight in $\mathbb{Q}_2^\times$ (see [22] for more on this).

Since also $\{x \in \mathbb{Q}_3 \mid \neg(\exists y)(y^2 = x)\}$ is definable, and therefore should be decomposable in cells (which we haven't defined yet), it is in some

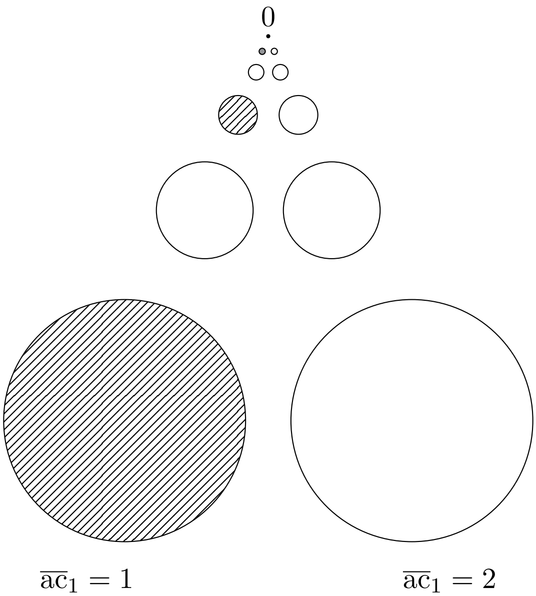


Figure 2.2: The squares in $\mathbb{Q}_3$

sense natural to write this set as

$$\{x \in \mathbb{Q}_3 \mid \overline{\mathrm{ac}}_1(x) = 2\} \cup \{x \in \mathbb{Q}_3 \mid \overline{\mathrm{ac}}_1(x) = 1, \mathrm{ord}(x) \in 1 + 2\mathbb{Z}\}.$$

We elaborate on this a bit. If we look again at Figure 2.2, we see that the set of dashed balls and its complement don't have the same kind of shape. In the set of dashed balls, there is at most one ball of each diameter, which will turn out to be a basic property of p-adic cells. However, in the complement of this set, namely the set of all non-dashed balls, this isn't true anymore. But after partitioning this set of non-dashed balls into the non-dashed balls on the left and the non-dashed balls on the right, this property will be valid again.

The condition $\mathrm{ord}(x) \in m+n\mathbb{Z}$ will therefore also appear in the definition of cells. Let us first verify that this is in fact a definable condition.

Lemma 9. *Let $c \in K$, let $n > 0$ be a positive integer, and let $m \in \{0, 1, \ldots, n-1\}$. Then $\{x \in K \mid \mathrm{ord}(x-c) \in m+n\mathbb{Z}\}$ is an $\mathcal{L}$-definable subset of K.*

Proof. Without loss of generality, we may assume that $c = 0$. We will use the fact that $P_n^\times$ is a subgroup of finite index in $K^\times$, see Corollary 6. Write

$$K^\times = \lambda_1 P_n^\times \cup \lambda_2 P_n^\times \cup \cdots \cup \lambda_s P_n^\times,$$

where the unions are disjoint, and let I_m be the set of indices i such that $\operatorname{ord}(\lambda_i) \equiv m \mod n$. It is then obvious that

$$\{x \in K \mid \operatorname{ord}(x) \in m + n\mathbb{Z}\} = \bigcup_{i \in I_m} \lambda_i P_n^\times. \qquad \square$$

We are now ready to give the definition of p-adic cells, based on [9] and [8]. For positive integers m and n, let $Q_{m,n}$ be the set

$$Q_{m,n} = \{x \in K^\times \mid \operatorname{ord}(x) \in n\mathbb{Z}, \overline{\operatorname{ac}}_m(x) = 1\}.$$

By Lemmas 9 and 8, the set $Q_{m,n}$ is definable.

Definition 10 (p-adic cell)**.** *Let Y be a definable set. A* cell $C \subset K \times Y$ *over Y is a (nonempty) set of the form*

$$C = \{(x, y) \in K \times Y \mid y \in Y',\ |\alpha(y)| \,\square_1\, |x - c(y)| \,\square_2\, |\beta(y)|,\ x - c(y) \in \xi Q_{m,n}\},$$

such that the following conditions hold: Y' is a definable subset of Y; $\xi \in K$; $\alpha, \beta : Y' \to K^\times$ and $c : Y' \to K$ are definable functions; $\square_i$ is either $<$ or "no condition"; and C projects surjectively onto Y'. One calls c and $\xi Q_{m,n}$ the center *and the* coset *of the cell C, respectively. If $\xi = 0$ one calls C a* 0-cell*, otherwise C is said to be a* 1-cell.

The definable set Y' in the definition of a cell acts as a *parameter space*, and is also called the *base* of the cell. In this way, one could view a cell as a *definable family* of 1-dimensional cells. This is why the name *cylindrical cell* is also used, to indicate that the cell-like description occurs only in one variable. Later, we will often study functions $f : X \times Y \to K$, where X is a definable subset of K and Y is a definable set of any dimension. Those functions could also be thought of as definable families of functions, where Y acts as the parameter space of this family.

Remark that originally, in [14], [15], and [16], p-adic cells were defined using the predicates P_n instead of $Q_{m,n}$. The following two lemmas show that this doesn't change anything about the power of cell-decomposition theorems in this context.

Lemma 11. *Let n be a positive integer, then there exists $m > 0$ such that $P_n^\times$ is a finite disjoint union of cosets of $Q_{m,n}$.*

Proof. Let $x \in P_n^\times$ and let m be such that $1 + \pi^m \mathcal{O}_K \subset P_n^\times$, see Corollary 4, then automatically $Q_{m,n} \subset P_n^\times$. Let $\xi \in \mathcal{O}_K^\times$ such that $\overline{\mathrm{ac}}_m(x) = \overline{\mathrm{ac}}_m(\xi)$. Then we can write $x = \xi y$, with $y \in Q_{m,n}$. This implies that $\xi \in P_n^\times$ as well. Then obviously $\xi Q_{m,n} \subset P_n^\times$. Now let $\{\xi_1, \xi_2, \ldots, \xi_s\} \subset \mathcal{O}_K^\times$ be a finite set of representatives of angular components of depth m of elements of $P_n^\times$. Then $P_n^\times$ is the disjoint union of the cosets $\xi_i Q_{m,n}$, where $i = 1, \ldots, s$. □

Corollary 12. *The multiplicative group $Q_{m,n}$ has finite index in $K^\times$.*

Proof. This follows immediately from the previous lemma and the fact that the nonzero n-th powers have a finite index in $K^\times$. □

Lemma 13. *Let m and n be positive integers, then there exists a positive integer N such that $Q_{m,n}$ is a finite disjoint union of cosets of $P_N^\times$.*

Proof. Let N' be the order of the multiplicative group $(\mathcal{O}_K/\pi^m\mathcal{O}_K)^\times$ and let $N = N' \cdot n$. We claim that $P_N^\times \subset Q_{m,n}$. Let therefore $x \in P_N^\times$, then we can write $x = y^N$ for some nonzero $y \in K^\times$, hence

$$\begin{aligned}\overline{\mathrm{ac}}_m(x) &= \overline{\mathrm{ac}}_m(y^N)\\ &= (\overline{\mathrm{ac}}_m(y)^{N'})^n\\ &\equiv 1 \mod \pi^m.\end{aligned}$$

Also the order of x is clearly a multiple of n, which proves the claim. Now conclude by nothing that $P_N^\times$ is a subgroup of $Q_{m,n}$, and since $P_N^\times$ has finite index in $K^\times$, it also has finite index in $Q_{m,n}$. □

The advantage of the $Q_{m,n}$ predicates is that one has the following nice description of cells, due to [8].

Proposition-Definition 14. *Let Y be a definable set. Let $C \subset K \times Y$ be a 1-cell over Y with center c and coset $\xi Q_{m,n}$. Then, for each $(t, y) \in C$ with $y \in Y$, there exists a unique maximal ball B containing t and* *satisfying* $B \times \{y\} \subset C$*, where maximality is with respect to inclusion. If* $\mathrm{ord}(t - c(y)) = l$*, this ball is of the form*

$$B = B_{l,c(y),m,\xi} = \{x \in K \mid \mathrm{ord}(x - c(y)) = l,\, \overline{\mathrm{ac}}_m(x - c(y)) = \overline{\mathrm{ac}}_m(\xi)\}.$$

One calls the collection of all these maximal balls the balls of the cell C*. For fixed* $y_0 \in Y$*, one calls the collection of balls*

$$\{B_{l,c(y_0),m,\xi} \mid B_{l,c(y_0),m,\xi} \times \{y_0\} \subset C\}$$

the balls of the cell C above y_0*. If* $C \subset K \times Y$ *is a* 0*-cell, one defines the collection of balls of* C *to be the empty collection.*

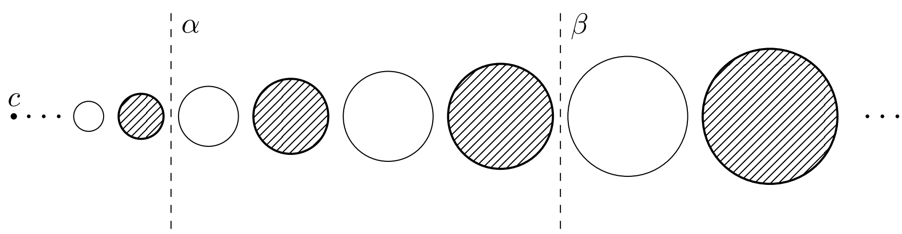


Figure 2.3: A p-adic 1-cell in dimension one

We now formulate a basic form of the p-adic cell decomposition theorem, due to Cohen ([14]) and Denef ([15, 16]) in the semi-algebraic case, and to Cluckers ([7]) in the subanalytic case. Usually, a cell decomposition theorem consist of two statements, one about definable sets, and the other about definable functions on these sets. We will for now only give the first statement, and discuss definable functions at a more suitable moment (see Theorems 18 and 26).

Theorem 15 (p-adic cell decomposition)**.** *Let* Y *and* $X \subset K \times Y$ *be definable sets. Then* X *can be decomposed as a finite disjoint union of* p*-adic cells over* Y*.*

The philosophy is the following: Lemmas 7, 8 and 9 show that every p-adic cell is $\mathcal{L}$-definable; Theorem 15 states that up to a finite partition, the converse is also true.

2.2.2 Tame geometry

Definable functions are expected to be *nice* and demonstrate some sort of *tame* (as opposed to *wild*) behavior. Indeed, since only a finite amount of data is used to define a definable function, nice properties might be expected from it, from a philosophical point of view. Often, one restricts to certain kinds of structures to obtain even more tame behavior. For example, in an o-minimal structure $\mathcal{R}$ (see example 1 on page 33), definable functions show this tame behavior through several properties concerning continuity and differentiability (see [19]):

1. Monotonicity: let $f : (a, b) \to R$ be a definable function on the open interval (a, b), then there are points $a_1, \ldots, a_{k-1} \in (a, b)$, such that on each subinterval (a_i, a_{i+1}), with $a = a_0$ and $b = a_k$, the function f is either constant, or strictly monotone (i.e. strictly increasing or decreasing) and continuous.
2. Differentiability: let $f : (a, b) \to R$ be a definable function on the open interval (a, b), then f is differentiable on a cofinite subset of (a, b).
3. Mean Value Theorem: let $f : [a, b] \to R$ be a definable and continuous function on the closed interval $[a, b]$, and differentiable on each point of (a, b), then for some $c \in (a, b)$, one has $f(b)-f(a) = f'(c)(b - a)$.

In [23], a p-adic analogue of o-minimality was proposed, namely P-minimality (see also example 2 on page 33). With the tame behavior of o-minimal functions in mind, one could expect a similar tame behavior of P-minimal functions (i.e. functions that are definable in a P-minimal structure).

To see whether p-adic analogues of the above properties hold in P-minimal structures, one first needs a good analogue in the p-adic setting of an interval. Since there is no ordering on p-adic fields, one needs to mimic other properties of the interval. For $x, y \in K$, one therefore lets (x, y) denote the smallest open ball containing both x and y, and this will act as the p-adic analogue of the interval (the definition of $[x, y]$ is similar, but now with closed balls). One says that z lies *between* x and y if z is contained in (x, y). Furthermore, one says that $f : K \to K$ is *monotone* if, whenever z lies between x and y, then also $f(z)$ lies between $f(x)$ and $f(y)$. Using this terminology, we prove a p-adic local version of the Monotonicity Theorem (see Theorem 48); a detailed exposition can be found in chapter 4.

We also obtain a differentiability result in the P-minimal setting: every definable function $f : X \subset K \to K$ is differentiable on a cofinite subset of X (see Theorem 46 for the precise statement).

There is, however, something unusual going on when it comes to differentiation in the non-Archimedean world. Consider, for example, the function

$$f : \mathbb{Z}_p \to \mathbb{Z}_p : \sum_{i=0}^{\infty} a_i p^i \mapsto \sum_{i=0}^{\infty} a_i p^{2i}. \tag{2.1}$$

It's clear from the definition that f is injective, but it's not hard to see that $f' = 0$ identically (see for example [25]). This is yet another striking example of the difference between the real world and the p-adic, since in the real world, if $f' = 0$ everywhere, this would imply that f is constant.

It should be noted, though, that the function f in (2.1) is not definable in a P-minimal structure. Indeed, first one notes that the image of f cannot contain any ball, since for any point x in the image of f, one can find points arbitrarily close to x with odd powers of p in their p-adic representation, and hence don't lie in the image of f. Since f is injective, the image does, however, contain infinitely many points. Therefore f cannot be P-minimal, because the image of f is a definable subset of $\mathbb{Q}_p$ that isn't the finite union of open sets and points.

Before one can check the Mean Value Theorem in the p-adic case, one should find a decent translation of it to the non-Archimedean context. This isn't hard with the notions introduced above: let $f : [x, y] \to K$ be a continuous function on $[x, y]$ (i.e. the smallest closed ball containing x and y), and differentiable on each point of (x, y), then f is said to satisfy the Mean Value Theorem if there is a z between x and y, such that $f(y) - f(x) = f'(z)(y - x)$.

The function in (2.1) makes it immediately clear that the Mean Value Theorem in general doesn't hold in the p-adic setting. Indeed, for any

$x \neq y$ in $\mathbb{Z}_p$, one has $f(x) \neq f(y)$, so one can never have $f(y) - f(x) = f'(z)(y - x)$, since $f'(z) = 0$ for all $z \in \mathbb{Z}_p$.

As we saw above, this function f is not P-minimal, so one could be tempted to think that maybe for definable functions in a P-minimal structure, the Mean Value Theorem *does* hold. Unfortunately, also this is not the case. To see this, consider the definable function

$$f : \mathbb{Z}_p \to \mathbb{Q}_p : x \mapsto x^p - x.$$

Then $f(0) = f(1) = 0$, but $f'(x) = px^{p-1} - 1 \neq 0$ for every $x \in \mathbb{Z}_p$, so f doesn't satisfy the Mean Value Theorem.

In an attempt to remedy this absence of the Mean Value Theorem in the p-adic setting, one introduces the following notion (see [8] or [9]).

Definition 16 (Jacobian Property)**.** *Let $f : B_1 \to B_2$ be a function, with $B_1, B_2 \subset K$. Say that f has the* Jacobian Property *if the following conditions hold:*

1. *B_1 and B_2 are balls and $f : B_1 \to B_2$ is a bijection;*
2. *f is continuously differentiable on B_1, with derivative f';*
3. $\mathrm{ord}(f')$ *is constant and finite on B_1;*
4. *for all $x, y \in B_1$ with $x \neq y$, one has:*
$$\mathrm{ord}(f(x) - f(y)) = \mathrm{ord}(f') + \mathrm{ord}(x - y). \tag{2.2}$$

Remark that (2.2) is equivalent to

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'|,$$

which would also follow from the Mean Value Theorem (when in the right hand side one would write $|f'(z)|$, for a suitable z). In this way, the Jacobian Property has a consequence of the Mean Value Theorem *built into* its definition.

A first step towards a tameness result of p-adic functions is the following lemma, which can be found in [8]. Recall that definable means either semi-algebraic or subanalytic.

Lemma 17. *Let Y and $X \subset K \times Y$ be definable sets and let $f : X \to K$ be a definable function. Then there exists a finite partition of X into definable sets X_i, such that for each $y \in Y$, the restriction of $f(\cdot, y) : x \mapsto f(x, y)$ to*

$$X_{iy} = \{x \in K \mid (x, y) \in X_i\}$$

is either injective or constant, and this distinction only depends on i (and not on y).

On the parts in the previous lemma where $f(\cdot, y)$ is injective, one has the following refinement due to [8], which comes close to what one could call a p-adic version of the Mean Value Theorem:

Theorem 18. *Let Y and $X \subset K \times Y$ be definable sets and let $f : X \to K$ be a definable function. Suppose that for each $y \in Y$, the function $f(\cdot, y) : x \mapsto f(x, y)$ is injective. Define $f \times \mathrm{id}$ as the (definable) function*

$$f \times \mathrm{id} : X \to K \times Y : (x, y) \mapsto (f(x, y), y).$$

Then there exists a finite partition of X into p-adic cells C_i over Y, such that for each i, the image $(f \times \mathrm{id})(C_i)$ is also a p-adic cell over Y, which we will denote by C_i'. Moreover, for each $y \in Y$ and each ball B of C_i above y, there is a ball B' of C_i' above y such that

$$f(\cdot, y)\big|_B : B \to B' : x \mapsto f(x, y)$$

is well defined and has the Jacobian Property.

Chapter 3

Lipschitz extensions of definable p-adic functions

This chapter is based on an article that will be published in the Mathematical Logic Quarterly, see [28]. The content of this chapter coincides with that of the article, but we elaborate more on some aspects. For example, there is more emphasis on the historical context of Lipschitz extensions, and we go deeper into some details of the proofs.

In the first section, we overview the history of the Lipschitz extension problem in the real and p-adic context. Also, we introduce some basic definitions and facts. In the second section, we prove the main theorems.

3.1 Historical overview and preliminary definitions

In this section we overview the history and the basic facts about the *Lipschitz extension problem.* Let us begin with the definition of a Lipschitz function and a Lipschitz extension.

Definition 19. *Let (F_1, d_1) and (F_2, d_2) be two metric spaces, where d_i denotes the metric on F_i, for $i = 1, 2$. Say a function $f : F_1 \to F_2$ is λ-Lipschitz (or λ-Lipschitz* continuous*), where λ is a positive real* *number, if the following holds for all $x, y \in F_1$:*

$$d_2(f(x), f(y)) \leq \lambda d_1(x, y).$$

Lipschitz continuity is a very strong form of continuity, since every Lipschitz function is uniformly continuous (the reverse is not true).

Given a λ-Lipschitz function $f : S \subset F_1 \to F_2$ that is only defined on a subset of F_1, one could wonder whether f could be *extended* to a λ-Lipschitz function $\tilde{f} : F_1 \to F_2$.

Definition 20. *Let F_1 and F_2 be as above, and let $f : S \subset F_1 \to F_2$ be a function. Say $\tilde{f} : F_1 \to F_2$ is a λ-*Lipschitz extension* of f, if $\tilde{f}$ is a λ-Lipschitz function that agrees with f on S, i.e. $\tilde{f}\big|_S = f$.*

The *Lipschitz extension problem* is nothing more than the question which functions allow Lipschitz extensions.

3.1.1 History

The starting point of our story is the year 1934, when Kirszbraun proved that every λ-Lipschitz function $f : S \subset \mathbb{R}^r \to \mathbb{R}^s$ extends to a λ-Lipschitz function $\tilde{f} : \mathbb{R}^r \to \mathbb{R}^s$ (see [26]). For $r = s = 1$, this isn't hard to prove; an example is illustrated in Figure 3.1.

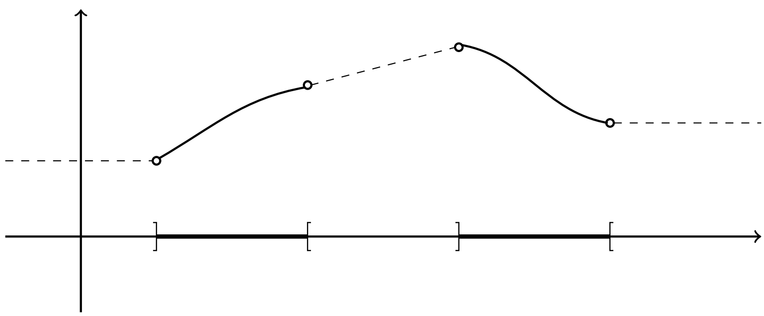


Figure 3.1: Lipschitz extension in the real case in dimension 1

The situation in this example is the following: the domain of f consists of two open intervals, on which f is 1-Lipschitz. On the left towards

$-\infty$ and on the right towards $+\infty$, one extends f constantly. Between the two intervals, one uses linear interpolation to extend f. Obviously, the resulting extension (of which the graph is indicated with a dashed line) will still be 1-Lipschitz. For functions $f : S \subset \mathbb{R} \to \mathbb{R}$ with a more complicated domain S, a similar construction can be done. For higher dimensional functions, however, the situation is more complicated. For one thing, the axiom of choice is needed for constructing Lipschitz extensions (see [1] for more details).

In 1983, Bhaskaran proved that a version of Kirszbraun's theorem still holds in a non-Archimedean setting (see [3]). We illustrate Bhaskaran's construction for a 1-Lipschitz function $f : S \subset K \to K$, where K is a p-adic field.

The idea is first to prove that one can always extend f to a 1-Lipschitz function $\tilde{f}_1 : S \cup \{x\} \to K$, where $x \in K \setminus S$. In subsection 3.1.2, we will see that every λ-Lipschitz function extends uniquely to the topological closure of its domain (see Lemma 22), so one may assume that S is already topologically closed. Of course, one defines $\tilde{f}_1(t) = t$ for all $t \in S$, so that $\tilde{f}_1$ agrees with f on S. Hence, one only needs to define $\tilde{f}_1(x)$. For this, choose any element a in S that lies closest to x (this is possible because S is topologically closed). Then define $\tilde{f}_1(x) = f(a)$. To verify that $\tilde{f}_1$ is 1-Lipschitz, one only needs to check that $|\tilde{f}_1(x) - \tilde{f}_1(s)| \leq |x - s|$ for every $s \in S$ (see Figure 3.2 for an overview of this setup).

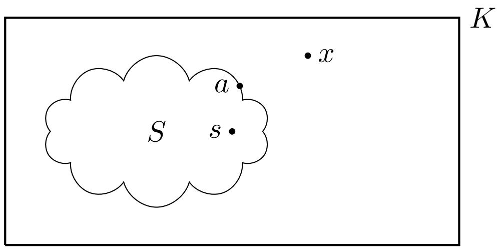


Figure 3.2: The extension $\tilde{f}_1$ of f to $S \cup \{x\}$

For this one calculates as follows:

$$\begin{aligned} |\tilde{f}_1(x) - \tilde{f}_1(s)| &= |f(a) - f(s)| \\ &\leq |a - s| \end{aligned}$$

$$\leq |x - s|,$$

where the last inequality holds by the following argument: if $|x - a| < |x - s|$, then by the non-Archimedean property one must have that $|a - s| = |x - s|$; if $|x - a| = |x - s|$, then again by the non-Archimedean property one has that $|a - s| \leq |x - s|$.

Now one constructs a 1-Lipschitz extension $\tilde{f} : K \to K$ of f as follows. Let L be the set of all 1-Lipschitz functions extending f. One defines a partial order on L by putting $g \leq h$ if and only if h extends g. Let $g_1 \leq g_2 \leq \cdots$ be any chain of elements of L, let V_i be the domain of g_i and let $V = \cup_i V_i$. Define $g : V \to K : x \mapsto g_i(x)$, where i is such that $x \in V_i$. Then g is well defined and 1-Lipschitz, hence g is an upper bound of the chain $g_1 \leq g_2 \leq \cdots$. Applying Zorn's Lemma then gives a maximal element $\tilde{f}$ of L. Let S be the domain of $\tilde{f}$, then we claim that $S = K$, so that $\tilde{f} : K \to K$ is the 1-Lipschitz extension of f that we were looking for. Suppose the claim were false, then there would exist an element $x \in K \setminus S$, and using the extension described above, one could extend $\tilde{f}$ to a 1-Lipschitz function on $S \cup \{x\}$, contradicting the maximality of $\tilde{f}$.

As was the case for real functions, also for p-adic functions Zorn's lemma appears to be an essential ingredient for the construction of Lipschitz extensions. One could wonder, however, if one would start from a nice category of functions, whether then also the Lipschitz extension would belong to this nice category. For this nice category of functions, one could for example consider the definable functions in a given structure. From a philosophical point of view, it is a somewhat *perverse* idea that in order to construct a Lipschitz extension of a *definable* Lipschitz function, one would use the axiom of choice. Indeed, such an extension would then be far from definable itself.

Recently, in 2010, Aschenbrenner and Fischer proved that, indeed, one can do much better: they proved a definable version of Kirszbraun's theorem. In particular, they proved that every λ-Lipschitz function $f : S \subset \mathbb{R}^r \to \mathbb{R}^s$, that is definable in an o-minimal expansion of the ordered field of real numbers, extends to a λ-Lipschitz function $\tilde{f} : \mathbb{R}^r \to \mathbb{R}^s$ that is definable in the same structure (see [1]). For $r = s = 1$ the picture is now *exactly* as in Figure 3.1, since in an o-minimal structure every definable subset of $\mathbb{R}$, and in particular the domain of every definable function $f : S \subset \mathbb{R} \to \mathbb{R}$, is a finite union of open intervals and points. Also, one immediately sees that the construction in Figure 3.1 is definable.

This leaves us, for now, with the following situation regarding the Lipschitz extension problem in the real and p-adic setting:

	real	***p*-adic**
non-definable	Kirszbraun (1934)	Bhaskaran (1983)
definable	Aschenbrenner-Fischer (2010)	

To fill the void on the definable p-adic side, Aschenbrenner posed the question whether there could be a *definable* version of Kirszbraun's theorem in a p-adic setting. In section 3.2 we partially answer this question and prove a definable version of Kirszbraun's theorem in a p-adic setting for definable families of functions *in one variable*. More precisely, we prove that every definable function $f : X \times Y \to K^s$, where $X \subset K$ and $Y \subset K^r$, that is λ-Lipschitz in the first variable, extends to a definable function $\tilde{f} : K \times Y \to K^s$ that is λ-Lipschitz in the first variable. By *definable*, we mean definable in either a semi-algebraic or a subanalytic structure on K. Working with these structures will allow us to use a cell decomposition result (see Theorem 26) that is essential for our construction of Lipschitz extensions.

3.1.2 Basic definitions and facts

Let us fix some notation and recall some facts from the previous chapters. In all that follows, K will always denote a p-adic field, i.e. a finite field extension of $\mathbb{Q}_p$. Denote by $\mathrm{ord} : K^\times \to \mathbb{Z}$ the valuation that is normalized such that the value group is $\mathbb{Z}$ (see subsection 1.2.2). Let q be the cardinality of the residue field of K. The valuation induces a non-Archimedean absolute value on K by setting $|x| = q^{-\mathrm{ord}(x)}$ for nonzero x, and $|0| = 0$. This extends to a non-Archimedean norm on K^s by setting $|(x_1, \ldots, x_s)| = \max_i\{|x_i|\}$.

The use of the max-norm has a useful consequence for Lipschitz-extensions:

Lemma 21. *If every λ-Lipschitz function $f : S \subset K^r \to K$ extends to a λ-Lipschitz function $\tilde{f} : K^r \to K$, then also every λ-Lipschitz function $f : S \subset K^r \to K^s$ extends to a λ-Lipschitz function $\tilde{f} : K^r \to K^s$.*

Proof. Let $f : S \subset K^r \to K^s$ be a λ-Lipschitz function given by

$$f(x) = (f_1(x), \dots, f_s(x)),$$

where $f_i : S \subset K^r \to K$ for each $i = 1, \dots, s$. Then for each $i = 1, \dots, s$, the following holds:

$$\begin{aligned}
|f_i(x) - f_i(y)| &\leq \max_j \{|f_j(x) - f_j(y)|\} \\
&= |(f_1(x), \dots, f_s(x)) - (f_1(y), \dots, f_s(y))| \\
&= |f(x) - f(y)| \\
&\leq \lambda |x - y|,
\end{aligned}$$

so f_i is λ-Lipschitz for each $i = 1, \dots, s$. Now let $\tilde{f}_i : K^r \to K$ be a λ-Lipschitz extension of f_i, which exists by assumption. Then

$$\tilde{f} : K^r \to K^s : x \mapsto (\tilde{f}_1(x), \dots, \tilde{f}_s(x))$$

is a λ-Lipschitz extension of f. Indeed, for $x, y \in K^r$, one has

$$\begin{aligned}
|\tilde{f}(x) - \tilde{f}(y)| &= |(\tilde{f}_1(x), \dots, \tilde{f}_s(x)) - (\tilde{f}_1(y), \dots, \tilde{f}_s(y))| \\
&= \max_i \{|\tilde{f}_i(x) - \tilde{f}_i(y)|\} \\
&\leq \lambda |x - y|.
\end{aligned}$$

□

This lemma will often be used to reduce to the case that the dimension of the target space is 1 (see e.g. the proofs of Theorems 29 and 34). Note that the previous lemma stays valid in a definable context.

The following lemma will allow us to reduce to the case that the domain of f is topologically closed.

Lemma 22. *Every definable and λ-Lipschitz function $f : S \subset K^r \to K^s$ extends uniquely to a definable and λ-Lipschitz function $\tilde{f} : \overline{S} \to K^s$, where $\overline{S}$ is the topological closure of S.*

Proof. Every λ-Lipschitz function $f : S \subset K^r \to K^s$ extends uniquely to $\tilde{f} : \overline{S} \to K^s$, where $\tilde{f}$ is defined as follows. Let $x \in \overline{S}$ and let $(x_n)_{n\geq 1}$ be a sequence of elements in S converging to x, then because f is λ-Lipschitz, $(f(x_n))_{n\geq 0}$ is a Cauchy sequence, so it converges since K is complete. Moreover, this limit only depends on x, and not on the sequence $(x_n)_{n\geq 0}$. We then define $\tilde{f}(x) = \lim_{n\to\infty} f(x_n)$. We prove that $\tilde{f}$ is λ-Lipschitz and definable.

Let $x, y \in \overline{S}$ be distinct elements. If $\tilde{f}(x) = \tilde{f}(y)$, there is nothing to prove, so we may assume that $\lim_{n\to\infty}(f(x_n) - f(y_n)) \neq 0$. Then there exists $N > 0$ such that $|\lim_{n\to\infty}(f(x_n) - f(y_n))| = |f(x_N) - f(y_N)|$. Increasing N if necessary, we may also assume that $|x_N - y_N| = |x - y|$. Then λ-Lipschitz continuity of $\tilde{f}$ follows easily:

$$
\begin{aligned}
|\tilde{f}(x) - \tilde{f}(y)| &= |\lim_{n\to\infty}(f(x_n) - f(y_n))| \\
&= |f(x_N) - f(y_N)| \\
&\leq \lambda |x_N - y_N| \\
&= \lambda |x - y|.
\end{aligned}
$$

Furthermore, $\tilde{f}$ is definable by the following formula:

$$\tilde{f}(x) = y \iff (\forall z \neq 0)(\exists (u, v) \in \Gamma(f))(|(u, v) - (x, y)| < |z|),$$

where x, y, u, and v denote multiple variables, $\Gamma(f)$ denotes the (definable) graph of f and where we use the following abbreviation: let D be a definable set, then by $(\exists t \in D)(\varphi(t))$ we mean $(\exists t)(t \in D \wedge \varphi(t))$. Moreover, the formula $|(u, v) - (x, y)| < |z|$ is short for

$$(\wedge_{i=1}^{r} |u_i - x_i| < |z|) \wedge (\wedge_{j=1}^{s} |v_j - y_j| < |z|),$$

expressing the fact that we are working with the max-norm. $\square$

Recall the following notation for balls of a p-adic cell (see Proposition-Definition 14):

$$B_{l,c(y),m,\xi} = \{x \in K \mid \operatorname{ord}(x - c(y)) = l, \, \overline{\operatorname{ac}}_m(x - c(y)) = \overline{\operatorname{ac}}_m(\xi)\}.$$

Notice that $B_{l,c(y),m,\xi}$ is a ball of diameter $q^{-(l+m)}$, so in particular, for every $x_1, x_2 \in B_{l,c(y),m,\xi}$ it holds that $|x_1 - x_2| \leq q^{-(l+m)}$.

We make the following notational conventions, which we will use throughout section 3.2.

Definition 23. *Let $f : S \subset K \times Y \to K$ be a function. Then we define*

$$f \times \operatorname{id} : S \to K \times Y : (x, y) \mapsto (f(x, y), y),$$

and we denote with S_f the image of $f \times \operatorname{id}$.

Definition 24. *Let $f : S \subset K \times Y \to K^s$ be a function. Then we define for every $y \in Y$*

$$f_y : S_y \to K^s : x \mapsto f(x, y),$$

where S_y denotes the fiber $S_y = \{x \in K \mid (x, y) \in S\}$.

The following notion from [9] is a refinement of the Jacobian Property (see Theorem 18).

Definition 25. *Let Y be a definable set, let $C \subset K \times Y$ be a 1-cell over Y, and let $f : C \to K$ be a definable function. Say that f is* compatible *with the cell C if either C_f is a 0-cell over Y, or the following holds: C_f is a 1-cell over Y and for each $y \in Y$, f_y is a bijection, and for each ball B of C above y and each ball B' of C_f above y, the functions $f_y|_B$ and $f_y^{-1}|_{B'}$ have the Jacobian Property.*

If $g : C \to K$ is a second definable function which is compatible with the cell C and if one has $C_f = C_g$ and $\operatorname{ord}(\frac{\partial f(x,y)}{\partial x}) = \operatorname{ord}(\frac{\partial g(x,y)}{\partial x})$ for every $(x, y) \in C$, then one says that f and g are equicompatible *with C.*

If $C' \subset K \times Y$ is a 0-cell over Y, any definable function $h : C' \to K$ is said to be compatible with C', and h and $k : C' \to K$ are equicompatible with C' if and only if $h = k$.

The following theorem is based on Theorem 3.3 of [9]. This theorem is the result of a constant refinement of the concept of p-adic cell decomposition for semi-algebraic and subanalytic structures. Earlier versions are due to Cohen [14], Denef [15, 16], and Cluckers [7], and relate to the quantifier elimination results from Macintyre [31] and Denef-van den Dries [17].

Theorem 26. *Let $S \subset K \times Y$ and $f : S \to K$ be definable. Then there exists a finite partition of S into cells C over Y such that the restriction $f|_C$ is compatible with C for each cell C. Moreover, for each cell C there exists a definable function $m : C \to K$, a definable function $e : Y \to K$ and coprime integers a and b with $b > 0$, such that for all $(x, y) \in C$*

$$m(x, y)^b = e(y)(x - c(y))^a,$$

where c is the center of C, and such that if one writes c' for the center of C_f, one has that $g = m + c'$ and f are equicompatible with C (we use the conventions that $b = 1$ whenever $a = 0$, that $a = 0$ whenever C is a 0-cell, and that $0^0 = 1$).

Furthermore, if C and C_f are 1-cells, then for every $y \in Y$ one has that $f_y(B) = g_y(B)$ for every ball B of C above y, and the formula

$$\operatorname{ord}\left(\frac{\partial f(x, y)}{\partial x}\right) = \operatorname{ord}(e(y)^{1/b} q) + (q - 1)\operatorname{ord}(x - c(y)) \tag{3.1}$$

holds for all $(x, y) \in C$, where $q = a/b$ and where we use the convenient notation $\operatorname{ord}(t^{1/b}) = \operatorname{ord}(t)/b$, for $t \in K$ and $b > 0$ a positive integer.

Proof. The existence of a finite partition of S in cells C over Y, and for every such a cell C the existence of $g = m + c'$ such that f and g are equicompatible with C, follows immediately from Theorem 3.3 in [9].

Now assume that C and C_f are 1-cells. It is easy to see that $f_y(B) = g_y(B)$ for every $y \in Y$ and every ball B of C above y.

We now prove (3.1). Fix $(x, y) \in C$. Since f and g are equicompatible, we have $\operatorname{ord}(\frac{\partial f(x,y)}{\partial x}) = \operatorname{ord}(\frac{\partial g(x,y)}{\partial x})$, so we only need to prove that (3.1) holds for f replaced by g. For this, we first note that

$$\operatorname{ord}(g(x, y) - c'(y)) = [\operatorname{ord}(e(y)) + a \cdot \operatorname{ord}(x - c(y))]/b. \tag{3.2}$$

It is also immediate that

$$\operatorname{ord}\left(\frac{\partial([g(x,y)-c'(y)]^b)}{\partial x}\right) = \operatorname{ord}(e(y)a(x-c(y))^{a-1}). \tag{3.3}$$

On the other hand, by the chain rule, the left hand side of (3.3) also equals

$$\operatorname{ord}\left(\frac{\partial([g(x,y)-c'(y)]^b)}{\partial x}\right) = \operatorname{ord}\left(b[g(x,y)-c'(y)]^{b-1}\frac{\partial g(x,y)}{\partial x}\right). \tag{3.4}$$

Equating the right hand sides of (3.3) and (3.4), and using (3.2), one easily finds the required formula. Although the calculations are straightforward, we include them for the convenience of the reader. From (3.3) and (3.4), we find:

$$\begin{aligned}
&\operatorname{ord}(e(y)) + \operatorname{ord}(a) + (a-1)\operatorname{ord}(x-c(y)) \\
&\qquad = \operatorname{ord}(b) + (b-1)\operatorname{ord}(g(x,y)-c'(y)) + \operatorname{ord}\left(\frac{\partial g(x,y)}{\partial x}\right) \\
&\qquad = \operatorname{ord}(b) + (b-1)\frac{\operatorname{ord}(e(y)) + a\cdot\operatorname{ord}(x-c(y))}{b} + \operatorname{ord}\left(\frac{\partial g(x,y)}{\partial x}\right),
\end{aligned}$$

where the last equation follows from (3.2). Reorganizing the terms then gives:

$$\begin{aligned}
\operatorname{ord}\left(\frac{\partial g(x,y)}{\partial x}\right) &= \frac{1}{b}\operatorname{ord}(e(y)) + \operatorname{ord}(a/b) + \frac{a-b}{b}\operatorname{ord}(x-c(y)) \\
&= \operatorname{ord}(e(y)^{1/b}q) + (q-1)\operatorname{ord}(x-c(y)).
\end{aligned}$$

□

Comparing sizes of balls between which there is a function with the Jacobian Property, we obtain the following useful formula.

Lemma 27. *Let $f : B_{l,c(y),m,\xi} \to B_{l',c'(y),m',\xi'}$ be a function with the Jacobian Property. Then $l' + m' = \operatorname{ord}(df/dx) + l + m$.*

Proof. Let $x, y \in B_{l,c,m,\xi}$ be two points with maximal distance, i.e.

$$\operatorname{ord}(x-y) = l + m. \tag{3.5}$$

Since $f(x), f(y) \in B_{l',c',m',\xi'}$ we have

$$\operatorname{ord}(f(x) - f(y)) \geq l' + m'. \tag{3.6}$$

But from the Jacobian Property the following holds:

$$\operatorname{ord}(f(x) - f(y)) = \operatorname{ord}(x - y) + \operatorname{ord}(f'), \tag{3.7}$$

which leeds to

$$\begin{aligned} l + m + \operatorname{ord}(f') &\overset{(3.5)}{=} \operatorname{ord}(x - y) + \operatorname{ord}(f') \\ &\overset{(3.7)}{=} \operatorname{ord}(f(x) - f(y)) \\ &\overset{(3.6)}{\geq} l' + m'. \end{aligned} \tag{3.8}$$

Now take $x, y \in B_{l,c,m,\xi}$ such that $f(x)$ and $f(y)$ have maximal distance in $B_{l',c',m',\xi'}$, then

$$\operatorname{ord}(f(x) - f(y)) = l' + m'. \tag{3.9}$$

Since $x, y \in B_{l,c,m,\xi}$, we know that

$$\operatorname{ord}(x - y) \geq l + m. \tag{3.10}$$

From this we can calculate:

$$\begin{aligned} l' + m' &\overset{(3.9)}{=} \operatorname{ord}(f(x) - f(y)) \\ &\overset{(3.7)}{=} \operatorname{ord}(x - y) + \operatorname{ord}(f') \\ &\overset{(3.10)}{\geq} l + m + \operatorname{ord}(f'). \end{aligned} \tag{3.11}$$

Combining (3.8) and (3.11) results in the desired formula. □

Another useful fact for Lipschitz functions with the Jacobian Property is the following.

Lemma 28. *Let B_1 and B_2 be balls and let $f : B_1 \to B_2$ be a function with the Jacobian Property, then the following are equivalent:*

1. $\mathrm{ord}(f') \geq 0$;
2. $|f'| \leq 1$;
3. f *is* 1*-Lipschitz.*

Proof. This is obvious. $\square$

Note that for a real function something similar happens, but then without needing extra assumptions on f: let $f : [a,b] \to \mathbb{R}$ be a function that is differentiable on (a,b), then by the Mean Value Theorem one has:

$$|f'(c)| \leq 1, \text{ for all } c \in (a,b) \iff f \text{ is 1-Lipschitz.}$$

This relates to the discussion at the end of section 2.2.2.

3.2 Main results

In this section we proceed towards proving the existence of definable Lipschitz extensions of definable families of p-adic functions in one variable. Let us first formulate the main theorem of this chapter:

Theorem 29. *Let $Y \subset K^r$ and $X \subset K$ be definable sets and let $f : X \times Y \to K^s$ be a definable function that is λ-Lipschitz in the first variable. Then f extends to a definable function $\tilde{f} : K \times Y \to K^s$ that is λ-Lipschitz in the first variable, i.e. $\tilde{f}_y$ is λ-Lipschitz for every $y \in Y$.*

Remark 30. Theorem 29 is indeed a result about *definable families* of functions, since the function $f : X \times Y \to K^s : (x,y) \mapsto f(x,y)$ can be identified with the family of functions $\{f_y : X \to K^s : x \mapsto f(x,y)\}_{y \in Y}$. We call this family *definable*, since there is one single formula $\varphi(x,y)$ defining the entire family. For every fixed value $y_0 \in Y$, the formula $\varphi(x, y_0)$ defines the graph of exactly one member of this family. This is much more restrictive then considering a family of definable functions, because then every member of the family might be defined by a different formula.

Remark 31. By rescaling, it suffices to prove Theorem 29 for $\lambda = 1$. Also, since we use the max-norm on K^s, it is enough to prove the theorem for $s = 1$ (see Lemma 21).

In the first subsection, we present a very general way of *gluing* Lipschitz extensions of a given function to obtain a Lipschitz extension with a larger domain (this is Lemma 32). This is one of the key ingredients for our construction of Lipschitz extensions.

In the second subsection, given a definable function that is λ-Lipschitz in the first variable, we give a more easy construction to obtain a definable extension that is Λ-Lipschitz in the first variable, where Λ is possibly larger than λ (this is Theorem 34). Finally, using a more involved argument, we show that one can take Λ equal to λ (this is Theorem 29).

3.2.1 Gluing extensions

The idea of *gluing* extensions is the following. Say we are given a definable and Lipschitz function $f : S \subset K^r \to K^s$ that we would like to extend to a definable and Lipschitz function $\tilde{f} : K^r \to K^s$. We can use p-adic cell decomposition to partition S in a finite number of p-adic cells. Assume that we know how to extend $f|_C : C \to K^s$, for each cell C in this partition. Presumably this extension will *not* agree with f on other cells in this partition. The question is then whether we could *glue* these extensions to obtain the extension $\tilde{f}$. For this we will *cut up* K in definable pieces, and define $\tilde{f}$ depending on these pieces. Let us make this more concrete.

Lemma 32 (Gluing extensions)**.** *Let $X \subset K^r$ be a definable set and let $f : X \to K$ be a definable and λ-Lipschitz function. Let $X = \cup_{i=1}^k X_i$ be a finite covering of X by definable subsets X_i. Call $f_i = f|_{X_i} : X_i \to K$.*

If every f_i extends to a definable and Λ_i-Lipschitz map $\tilde{f}_i : K^r \to K$, with $\Lambda_i \geq \lambda$, then f extends to a definable and Λ-Lipschitz map $\tilde{f} : K^r \to K$, where $\Lambda = \max_i\{\Lambda_i\}$.

Proof. We prove the lemma first for $k = 2$. Define $T_1 = \{x \in K^r \mid d(x, X_1) \leq d(x, X_2)\}$, where $d(x, A)$ denotes the distance from x to the set A, i.e. $d(x, A) = \inf\{|x-a| \mid a \in A\}$. Define $T_2 = K^r \setminus T_1$, see Figure 3.3 for a visualization.

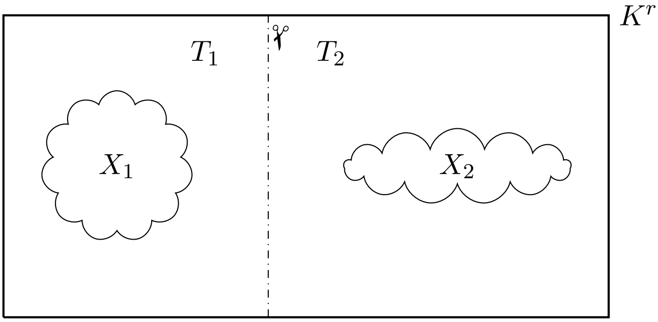


Figure 3.3: The sets T_1 and T_2

Let

$$\tilde{f} : K^r \to K : x \mapsto \begin{cases} \tilde{f}_1(x) & \text{if } x \in T_1, \\ \tilde{f}_2(x) & \text{if } x \in T_2. \end{cases}$$

Clearly, $\tilde{f}$ is a definable extension of f. We prove that $\tilde{f}$ is Λ-Lipschitz, where $\Lambda = \max\{\Lambda_1, \Lambda_2\}$. The only nontrivial fact to verify is that for $t_1 \in T_1$ and $t_2 \in T_2$, we have $|\tilde{f}(t_1) - \tilde{f}(t_2)| \leq \Lambda|t_1 - t_2|$.

Since every definable and λ-Lipschitz function extends uniquely to a definable and λ-Lipschitz function on the topological closure of its domain (this is Lemma 22), we may assume that X, X_1 and X_2 are topologically closed.

Fix elements $a_i \in X_i$ such that $|t_i - a_i| = d(t_i, X_i)$, for $i = 1, 2$. Then it always holds that

$$|t_2 - a_2| < |a_1 - a_2|. \tag{3.12}$$

We can now calculate as follows:

$$\begin{aligned} |\tilde{f}(t_1) - \tilde{f}(t_2)| &= |\tilde{f}_1(t_1) - f(a_1) + f(a_2) - \tilde{f}_2(t_2) + f(a_1) - f(a_2)| \\ &\leq \max\{|\tilde{f}_1(t_1) - f(a_1)|, |f(a_2) - \tilde{f}_2(t_2)|, |f(a_1) - f(a_2)|\} \\ &\leq \max\{\Lambda_1|t_1 - a_1|, \Lambda_2|a_2 - t_2|, \lambda|a_1 - a_2|\} \\ &\leq \max\{\Lambda_1, \Lambda_2\} \max\{|t_1 - a_1|, |a_2 - t_2|, |a_1 - a_2|\} \end{aligned}$$

$$\overset{(3.12)}{=} \Lambda \max\{|t_1 - a_1|, |a_1 - a_2|\}$$

$$\leq \Lambda |t_1 - t_2|,$$

where we only have to verify the last inequality. For this we prove that

$$\max\{|t_1 - a_1|, |a_1 - a_2|\} \leq |t_1 - t_2|, \tag{3.13}$$

by considering two cases.

Case 1: $|t_1 - a_1| < |a_1 - a_2|$**.** It then holds that

$$|a_1 - a_2| = |t_1 - a_2|. \tag{3.14}$$

So

$$|t_2 - a_2| \overset{(3.12)}{<} |a_1 - a_2| \overset{(3.14)}{=} |t_1 - a_2|, \tag{3.15}$$

hence

$$|a_1 - a_2| \overset{(3.14)}{=} |t_1 - a_2| \overset{(3.15)}{=} |t_1 - t_2|. \tag{3.16}$$

Case 2: $|a_1 - a_2| \leq |t_1 - a_1|$**.** Suppose that

$$|t_1 - t_2| < |t_1 - a_1|, \tag{3.17}$$

then

$$|t_1 - a_1| \overset{(3.17)}{=} |t_2 - a_1| \overset{(3.12)}{=} |a_1 - a_2|. \tag{3.18}$$

By the choice of a_1 and the fact that $t_1 \in T_1$, we know $|t_1 - a_2| \geq |t_1 - a_1|$, so by (3.18) equality holds:

$$|t_1 - a_1| = |t_1 - a_2|. \tag{3.19}$$

Together with (3.17), this implies

$$|t_1 - t_2| < |t_1 - a_2|. \tag{3.20}$$

So finally,

$$|a_1 - a_2| \overset{(3.18)}{=} |t_1 - a_1| \overset{(3.19)}{=} |t_1 - a_2| \overset{(3.20)}{=} |t_2 - a_2| \overset{(3.12)}{<} |a_1 - a_2|,$$

which is a contradiction.

This proves (3.13), and therefore the lemma is proved for $k = 2$. An easy induction argument then proves the lemma for general k. □

Remark 33. Lemma 32 remains true if one replaces every instance of the word “Lipschitz” by “Lipschitz in the first variable”.

3.2.2 Constructing Lipschitz extensions

Theorem 34. *Let $Y \subset K^r$ and $X \subset K$ be definable sets and let $f : S = X \times Y \to K^s$ be a definable function that is λ-Lipschitz in the first variable. Then there exists $\Lambda \geq \lambda$ such that f extends to a definable function $\tilde{f} : K \times Y \to K^s$ that is Λ-Lipschitz in the first variable, i.e. $\tilde{f}_y$ is Λ-Lipschitz for every $y \in Y$.*

Proof. By Remark 31, we may assume that $\lambda = 1$ and $s = 1$. By Theorem 26 and (the remark after) Lemma 32, we may assume that S is a cell over Y with which f is compatible. Furthermore, we may assume that the base of S is Y.

If S_f is a 0-cell over Y with center c', we define

$$\tilde{f} : K \times Y \to K : (x, y) \mapsto c'(y).$$

Clearly, $\tilde{f}$ is a definable extension of f and for all $y \in Y$, $\tilde{f}_y$ is 1-Lipschitz.

Assume from now on that S and S_f are 1-cells over Y, with center c and c', and coset $\xi Q_{m,n}$ and $\xi' Q_{m',n'}$, respectively.

We define $\tilde{f}$ as follows:

$$\tilde{f} : K \times Y \to K : (x, y) \mapsto \begin{cases} f(x, y) & \text{if } (x, y) \in S, \\ c'(y) & \text{if } (x, y) \notin S. \end{cases}$$

Clearly, $\tilde{f}$ is a definable extension of f. We prove that $\tilde{f}_y$ is $q^{m'}$-Lipschitz for every $y \in Y$.

Fix $y \in Y$. Let $t_1 \in X$ and $t_2 \notin X$. Let l and l' be such that $t_1 \in B_{l,c(y),m,\xi}$ and $f(t_1, y) \in B_{l',c'(y),m',\xi'}$. Then

$$\begin{aligned} |f(t_1, y) - c'(y)| &= q^{-l'} \\ &= q^{-\mathrm{ord}(\partial f(t_1,y)/\partial x)} q^{m'-m} q^{-\mathrm{ord}(t_1 - c(y))} \\ &\leq q^{m'-m} |t_1 - c(y)|, \end{aligned} \tag{3.21}$$

where the second equality follows from Lemma 27 and the last inequality holds because f is 1-Lipschitz in the first variable, and therefore $|\partial f(t_1, y)/\partial x| \leq 1$ (see Lemma 28). There are two cases to consider.

Case 1: $|t_1 - c(y)| = |t_2 - c(y)|$**.** Because $B_{l,c(y),m,\xi}$ is a ball of diameter q^{-m-l}, it holds that $|t_1 - t_2| > q^{-m-l}$, or put differently:

$$q^{-m}|t_1 - c(y)| < |t_1 - t_2|. \tag{3.22}$$

Therefore

$$\begin{aligned} |\tilde{f}_y(t_1) - \tilde{f}_y(t_2)| &= |\tilde{f}(t_1, y) - \tilde{f}(t_2, y)| \\ &= |f(t_1, y) - c'(y)| \\ &\overset{(3.21)}{\leq} q^{m'-m}|t_1 - c(y)| \\ &\overset{(3.22)}{<} q^{m'}|t_1 - t_2|. \end{aligned}$$

Case 2: $|t_1 - c(y)| \neq |t_2 - c(y)|$**.** From the non-Archimedean property it then follows that

$$|t_1 - c(y)| \leq |t_1 - t_2|, \tag{3.23}$$

so we find

$$\begin{aligned} |\tilde{f}_y(t_1) - \tilde{f}_y(t_2)| &= |\tilde{f}(t_1, y) - \tilde{f}(t_2, y)| \\ &= |f(t_1, y) - c'(y)| \\ &\overset{(3.21)}{\leq} q^{m'-m}|t_1 - c(y)| \\ &\overset{(3.23)}{\leq} q^{m'-m}|t_1 - t_2|. \end{aligned}$$ □

Remark 35*.* Analyzing the proof of Theorem 34, we find that one can take $\Lambda = \lambda \max_i\{q^{m'_i}\}$, where λ is the Lipschitz constant of f (in the first variable), and the m'_i correspond to the 1-cells in the cell decomposition of S_f.

Remark 36*.* We can even improve (i.e. decrease) Λ from Remark 35 as follows. In the proof, the worst Lipschitz constant occurs in **Case 1**. We can get around this case in the following way (as in the beginning of Theorem 34, we assume that S and S_f are 1-cells over Y and that S has center c and coset $\xi Q_{m,n}$).

For every nonzero $a \in \mathcal{O}_K^\times/(\pi_K^m)$, choose $\xi_m(a) \in \mathcal{O}_K^\times$ to be a class representative of a. Since we only need to make a finite number of representative choices, $\xi_m : \mathcal{O}_K^\times/(\pi_K^m) \to K^\times$ is a definable map. Let $\varphi : K \times Y \to K$ be the definable map rescaling the angular component as follows:

$$\varphi : K \times Y \to K :$$

$$(x, y) \mapsto \begin{cases} (x - c(y))\xi_m\big(\overline{\mathrm{ac}}_m(x - c(y))^{-1}\overline{\mathrm{ac}}_m(\xi)\big) + c(y) & \text{if } x \neq c(y), \\ c(y) & \text{if } x = c(y). \end{cases}$$

It is not difficult to see that for every $y \in Y$, $\varphi_y : K \to K$ is 1-Lipschitz. Now let $\hat{f}$ be the extension described in the proof of Theorem 34 in the case that S and S_f are 1-cells over Y (attention: in Theorem 34, this extension is denoted with $\tilde{f}$). Then $\tilde{f} : K \times Y \to K : (x, y) \mapsto \hat{f}(\varphi(x, y), y)$ is a definable extension of f that is $q^{m'-m}$-Lipschitz in the first variable. One can therefore take $\Lambda = \lambda \max_i\{q^{m_i'-m_i}\}$, where λ is the Lipschitz constant of f (in the first variable), and m_i and m_i' correspond to the 1-cells in the cell decomposition of S and S_f, respectively.

Note that in the proof of Theorem 34 we did not use the full generality of Theorem 26. We will now prove Theorem 29, the main theorem of this chapter, which uses a more involved extension for which the Lipschitz constant does not grow. For this, the full power of Theorem 26 is used. Again, the result is formulated for definable families of functions. For clarity, we repeat the formulation of Theorem 29.

Theorem. *Let $Y \subset K^r$ and $X \subset K$ be definable sets and let $S = X \times Y$. Let $f : S \to K^s$ be a definable function that is λ-Lipschitz in the first variable. Then f extends to a definable function $\tilde{f} : K \times Y \to K^s$ that is λ-Lipschitz in the first variable, i.e. $\tilde{f}_y$ is λ-Lipschitz for every $y \in Y$.*

Proof. By Remark 31, we may assume that $\lambda = 1$ and $s = 1$. By Theorem 26 and (the remark after) Lemma 32, we may assume that S is a cell over Y with which f is compatible. Furthermore, we may assume that the base of S is Y.

If S_f is a 0-cell, extend f as in Theorem 34.

Assume from now on that S and S_f are 1-cells over Y, with center c and c', and coset $\xi Q_{m,n}$ and $\xi' Q_{m',n'}$, respectively. Let g be as in Theorem 26, in particular f and g are equicompatible with S, and $(g(x,y) - c'(y))^b = e(y)(x - c(y))^a$ for every $(x,y) \in S$.

Fix $y \in Y$ and let $B_{l,c(y),m,\xi}$ be a ball of S above y. By Theorem 26 we can write $f_y(B_{l,c(y),m,\xi}) = B_{l',c'(y),m',\xi'} = g_y(B_{l,c(y),m,\xi})$, where $B_{l',c'(y),m',\xi'}$ is a ball of S_f above y. Also, we have that $\mathrm{ord}(\partial f/\partial x) = \mathrm{ord}(\partial g/\partial x)$. Let $q = a/b$, then there are three different cases to consider, depending on whether $q = 1$, $q < 1$ or $q > 1$.

Case 1: $q = 1$**.** From equation (3.1) on page 55, for all $(x,y) \in S$ it holds that $\mathrm{ord}(\partial f(x,y)/\partial x) = \mathrm{ord}(e(y))$. So for $x \in B_{l,c(y),m,\xi}$ we have

$$\begin{aligned} l' &= \mathrm{ord}(e(y)(x - c(y)) + c'(y) - c'(y)) \\ &= \mathrm{ord}(e(y)) + \mathrm{ord}(x - c(y)) \\ &= \mathrm{ord}(\partial f(x,y)/\partial x) + l, \end{aligned}$$

which implies $l' \geq l$, since f is 1-Lipschitz in the first variable (see Lemma 28). In particular note that in this case $m = m'$, by Lemma 27. This allows us to use the same extension as described in Remark 36, namely $\tilde{f} : K \times Y \to K : (x,y) \mapsto \hat{f}(\varphi_y(x), y)$, where $\hat{f}$ is as in the proof of Theorem 34 in the case that S and S_f are 1-cells over Y, and φ_y is as in Remark 36 (again, remark that in Theorem 34 this extension is denoted with $\tilde{f}$). We prove that $\tilde{f}_y$ is 1-Lipschitz. Let $t_1 \in \cup_l D_{l,c(y)}$ and $t_2 \notin \cup_l D_{l,c(y)}$, where $D_{l,c(y)} = \{x \in K \mid \mathrm{ord}(x - c(y)) = l\}$ and where the union $\cup_l D_{l,c(y)}$ runs over all l such that $B_{l,c(y),m,\xi} \subset S$. Then

$$\begin{aligned} |\tilde{f}_y(t_1) - \tilde{f}_y(t_2)| &= |f(\varphi_y(t_1), y) - c'(y)| \\ &\leq |\varphi_y(t_1) - c(y)| \\ &\leq |t_1 - t_2|, \end{aligned}$$

where the first inequality follows from $l' \geq l$ and the second from the non-Archimedean property.

Case 2: $q > 1$**.** Because f is 1-Lipschitz in the first variable, we have $\operatorname{ord}(\partial f/\partial x) \geq 0$ (see Lemma 28), and together with (3.1) this gives the following lower bound:

$$l \geq -\operatorname{ord}(e(y)^{1/b}q)/(q-1).$$

Recall that $\operatorname{ord}(e(y)^{1/b}q)$ is short for $\operatorname{ord}(e(y))/b + \operatorname{ord}(q)$. On the other hand, as soon as $l \geq (m' - m - \operatorname{ord}(e(y)^{1/b}q))/(q-1)$, we have $l' \geq l$. Indeed, this follows immediately from Lemma 27 and from (3.1). So up to partitioning S into two cells over Y, we may assume that either $l' \geq l$ for all balls of S above y, for every $y \in Y$, or that S has at most N balls above y, for every $y \in Y$, where N does not depend on y. In the former case we can extend f as we did in **Case 1**. In the latter case we can, after partitioning Y in a finite number of definable sets, assume that there are *exactly* N balls of S above y, for every $y \in Y$. Using (the remark after) Lemma 32 we may assume that there is exactly one ball of S above y, for every $y \in Y$. By definable selection (see [17] and [18]) there is a definable function $h : Y \to K$ such that for each $(x, y) \in S$ with $x \in K$ and $y \in Y$, $(h(y), y) \in S$. We then extend f as follows:

$$\tilde{f} : K \times Y \to K : (x,y) \mapsto \begin{cases} f(x,y) & \text{if } (x,y) \in S, \\ f(h(y),y) & \text{if } (x,y) \notin S. \end{cases}$$

Fix $y \in Y$, we show that $\tilde{f}_y$ is 1-Lipschitz. Recall that by the argument given above, S_y is a ball in K. The only nontrivial case to consider is the following. Let $t_1 \in S_y$ and $t_2 \notin S_y$, then

$$\begin{aligned} |\tilde{f}_y(t_1) - \tilde{f}_y(t_2)| &= |f(t_1, y) - f(h(y), y)| \\ &\leq |t_1 - h(y)| \\ &< |t_1 - t_2|, \end{aligned}$$

where the last inequality holds because of the non-Archimedean property and the fact that t_1 and $h(y)$ are both contained in the ball S_y, and t_2 is not.

Case 3: $q < 1$**.** This case is similar to **Case 2**, where now one finds an upper bound for l instead of a lower bound. The proof is omitted. □

Remark 37. Note that we proved the main theorem for semi-algebraic and subanalytic structures on K. In chapter 5 we generalize this result to a more general framework, including P-minimal structures endowed with definable selection. It is unclear whether the extension that we constructed could be used to extend a definable function $f : X \subset K^r \to K^s$ that is λ-Lipschitz in *all* variables to a definable function $\tilde{f} : K^r \to K^s$ that is λ-Lipschitz in all variables. In chapter 5 we focus some more on this and other questions.

Chapter 4

Differentiation in P-minimal structures

This chapter is a direct copy of the article [29], which will be published in the Journal of Symbolic Logic, and which is joint work with Eva Leenknegt. The original title of the article is *Differentiation in P-minimal structures and a p-adic Local Monotonicity Theorem.*

The aim of this research project was to answer an open question posed by Haskell and Macpherson in 1997 about a p-adic local monotonicity theorem for definable functions in P-minimal structures, more specifically Problem 7.6 of [23]. We found an answer to this question for a large class of P-minimal structures, and while doing so, we also solved another problem in the same article, namely Problem 7.5.

In the first section of this chapter, we formulate the open problem by Haskell and Macpherson, we introduce the tools that are used to attack the problem, and review some facts about P-minimal structures. In the second section we formulate and prove the main results, first for p-adic fields and later for strictly P-minimal structures.

4.1 Introduction

A major tool in the study of o-minimal structures is the Monotonicity Theorem, which states that for any o-minimal function $f : D \subset R \to R$, there exists a finite partition of D, such that on each part, f is either constant or continuous and strictly monotone (see e.g. van den Dries [19]).

When Haskell and Macpherson developed their theory of P-minimality [23] (a p-adic counterpart to the concept of o-minimality), a question that came up naturally was whether there would exist a p-adic version of this theorem. Of course, this question only makes sense if one can find a reasonable translation of the concept of *monotonicity* to the p-adic context.

Say that z lies between x and y if z is contained in the smallest ball that contains both x and y. Using this notion, one can formulate a concept of monotonicity that works both in the real and the p-adic setting:

Definition 38. *Let F be a metric field. A function $f : F \to F$ is* monotone *if, whenever z lies between x and y, then also $f(z)$ lies between $f(x)$ and $f(y)$.*

For ultrametric fields, this condition is equivalent to

$$|x - z| \leqslant |x - y| \Rightarrow |f(x) - f(z)| \leqslant |f(x) - f(y)|.$$

Observe that if a function f is monotone and $f(x) = f(y)$, then f is constant *between* x and y. A more detailed exploration can be found in [42].

Extending this idea further, the following would be a natural translation to the p-adic context of (local) strict monotonicity. (Remember that in the real case, a strictly monotone function f is either strictly increasing or decreasing, and hence we get a bijection between the domain of f and the image of f.)

Definition 39. *Let F be an ultrametric field. A function $f : X \subset F \to F$ is said to be* locally strictly monotone *on X if for all $a \in X$, there exist* *balls B_1, B_2 such that $a \in B_1 \subset X$, f maps B_1 bijectively onto B_2, and for all $x, y, z \in B_1$,*

$$|x - z| < |x - y| \Rightarrow |f(x) - f(z)| < |f(x) - f(y)|.$$

Having this notion in mind, Haskell and Macpherson [23] stated the following conjecture, which can be considered as a local version of the Monotonicity Theorem for p-adically closed fields K.

Conjecture 40. *Let $f : X \subset K \to K$ be a function definable in a P-minimal structure $(K, \mathcal{L})$. There exist definable disjoint subsets U, V of X, with $X \setminus (U \cup V)$ finite, such that*

(a) $f|_U$ is locally constant,

(b) $f|_V$ is locally strictly monotone.

Unfortunately, Haskell and Macpherson could only prove a weaker version of this conjecture. The main motivation of this paper is to give a full proof (we even obtain a slightly more precise result). The key to the problem is the existence of the first order (strict) derivative of P-minimal functions.

4.1.1 Differentiation in the *p*-adic context

Ever since the end of the 19th century, the theory of differentiation (and integration) of real functions has been well established. However, the picture is not quite as rosy when considering p-adic functions. Whereas real analysis has become a basic tool (even for non-mathematicians), p-adic analysis is more subtle for several reasons.

One of the consequences of the ultrametric topology is that the Mean Value Theorem no longer holds on p-adic fields. Because of this, even if we restrict to the category of *nice* functions that have a continuous derivative, examples can be found of functions that behave badly: an injective function that has a derivative which is zero everywhere, or a function that has nonzero derivative, yet is not injective in any neighbourhood of zero (see e.g. examples 26.4 and 26.6 in Schikhof's book [42].)

To remedy some of the problems listed above, we will need to consider a stronger concept of differentiation. A natural candidate is the following notion of strict differentiation (a detailed exposition of which can be found in Schikhof [42] or Robert [41]):

Definition 41. *Let $X \subset K$ be an open set. A function $f : X \to K$ is* strictly differentiable *at a point $a \in X$, with strict derivative $Df(a)$ if the limit*

$$Df(a) = \lim_{(x,y)\to(a,a)} \frac{f(x) - f(y)}{x - y}$$

exists.

To distinguish between both concepts, we use the notation Df to refer to the strict derivative, and we write f' for the normal derivative (as defined by Weierstrass). Obviously, $Df(a) = f'(a)$ whenever $Df(a)$ exists.

If f is strictly differentiable on an open set U, then Df is continuous on U, by Proposition 27.2 of [42], which means that a strictly differentiable function is automatically C^1.

Note that in the real case, every function f for which f' is continuous is automatically strictly differentiable, as can be seen easily by applying the Mean Value Theorem. One way to look at it is that strict differentiation is a form of continuous differentiation where (consequences of) the Mean Value Theorem are already built into the definition.

More recently, Bertram, Glöckner and Neeb [2] developed a more general framework for differential calculus. When restricted to functions of one variable over ultrametric fields, their notion is equivalent to strict differentiation (see Section 6 of [2] for a comparison). There are also some differentiability results in the C-minimal setting, in Section 5 of [24].

When using the stronger concept of strict differentiation, one can recover a number of the results that are foundational in real analysis. For example, a function that has nonzero strict derivative around a point x_0, will be injective on some neighborhood of x_0. However, some fundamental problems remain. For instance, the strictly differentiable function

$$g : \mathbb{Q}_p \to \mathbb{Q}_p : \sum_n a_n p^n \mapsto \sum_n a_n p^{2n} \tag{4.1}$$

is injective, and yet $Dg(x) = 0$ for all $x \in \mathbb{Q}_p$. This example shows that, even with a stronger concept of differentiation, a nice theory will only be achievable if one also restricts to a more tame class of functions. Note that something similar has been done for real functions as well. Indeed, it is known that o-minimal functions $f : \mathbb{R} \to \mathbb{R}$ are (continuously) differentiable on a cofinite subset of $\mathbb{R}$ (see van den Dries [19]).

Moreover, note that the function g from (4.1) is not P-minimal. Indeed, in P-minimal structures every infinite definable subset of the universe contains an open set. Clearly this is not true for $g(\mathbb{Q}_p)$.

4.1.2 Basic definitions and facts

Let us first review some basic facts about P-minimality (more details can be found in [23]).

Let $\mathcal{L}$ be a language extending the ring language $\mathcal{L}_{\text{ring}} = (+, -, \cdot, 0, 1)$, and let K be a p-adically closed field (that is, a field elementary equivalent, in the language of rings, to a finite field extension of $\mathbb{Q}_p$). The structure $(K, \mathcal{L})$ is said to be *P-minimal* if, for every elementary equivalent structure $(K', \mathcal{L})$, any definable set $X \subset K'$ is $\mathcal{L}_{\text{ring}}$-definable (with parameters from K').

Examples of P-minimal structures include p-adic semi-algebraic sets and the structure of p-adic subanalytic sets, as developed by Denef and van den Dries [17]. A function is said to be P-minimal if its graph is definable in a P-minimal structure.

By definable we always mean definable with parameters (and the underlying structure will be assumed to be P-minimal). A finite field extension of $\mathbb{Q}_p$ will also be called a p-adic field.

As a consequence, any definable subset of K can be partitioned into a finite number of points and a finite number of open sets. This implies that every infinite definable subset of K contains an open set, a fact which we will use quite often. The following lemma will also be used extensively.

Lemma 42 (Lemma 5.1 of [23]). *Let $f : X \subset K \to K$ be a function definable in a P-minimal structure $(K, \mathcal{L})$. There is a cofinite subset $U \subset X$ such that $f|_U$ is continuous.*

We also mention the following theorem, which is a corollary of Theorem 71.2 of [42]. This will be used to deduce strict differentiability from normal differentiability.

Theorem 43. *Let K be a complete non-Archimedean field and $X \subset K$ an open set. If $f : X \to K$ is differentiable, then the set*

$$\{x \in X \mid f \text{ is strictly differentiable at } x\}$$

is dense in X.

We write $\mathcal{O}_K$ for the valuation ring and Γ_K for the value group. Let π denote a fixed element with minimal positive valuation. Write $P_n^\times$ for the set of nonzero n-th powers in K, and $\lambda P_n^\times$ for the coset $\{\lambda x \mid x \in P_n^\times\}$, where $\lambda \in K$. Since $P_n^\times$ has finite index in $K^\times$, one can choose a finite subset $\Lambda_n \subset K^\times$ such that $K^\times = \cup_{\lambda \in \Lambda_n} \lambda P_n^\times$.

We let $B(x_0, \delta)$ denote the open ball with center x_0 and radius δ, i.e.

$$B(x_0, \delta) = \{x \in K \mid |x - x_0| < \delta\}.$$

We write $|K| = \{|x| \mid x \in K\}$. The notation $|f'(x_0)| = +\infty$ means that

$$\lim_{t \to 0} \left| \frac{f(x_0 + t) - f(x_0)}{t} \right| = +\infty.$$

4.1.3 Main results

We cannot formulate our results for P-minimal structures in general. The main reason for this restriction is the following lemma, which will be essential.

Lemma 44. *Let K be a p-adic field and let $f : X \subset K \to K$ be a differentiable function that is definable in a P-minimal structure. If $f'(x) = 0$ for all $x \in X$, then there exists a finite partition of X in parts X_i such that $f|_{X_i}$ is constant.*

A similar result can be found in [42]. A further generalisation to the context of p-adic integration is given in Proposition 64. For real functions with connected domain, this is a simple consequence of the Mean Value Theorem. However, this does not hold for general p-adic functions: the function g defined in (4.1) provides a counterexample.

In our proof of Lemma 44 we use that every open cover of K has a countable subcover, hence the extra condition that K is a p-adic field. We do not know whether this condition is essential.

We will show that our main results hold for any P-minimal structure satisfying the following additional condition:

Definition 45. *A P-minimal structure $(K, \mathcal{L})$ is said to be* strictly P-minimal *if there exists a finite field extension K' of $\mathbb{Q}_p$, such that $(K', \mathcal{L})$ is P-minimal, and K and K' are elementarily equivalent as $\mathcal{L}$-structures.*

Note that if Lemma 44 would be true for all p-adically closed fields, then the condition of *strict P-minimality* could be replaced by *P-minimality* for all subsequent results.

When working with general p-adically closed fields, it may happen that the value group $|K^\times|$ (considered as a multiplicative group) is not contained in $\mathbb{R}^\times$. In this case the limit of a function (and the derivative) can still be defined by the usual (ϵ, δ)-definition, the only difference being that ϵ and δ will be elements of $|K|$ rather than $\mathbb{R}$.

Another new result (see Proposition 54) that is crucial to our proofs is the fact that $\dim(\overline{X} \setminus X) < \dim(X)$ for any set X definable in a P-minimal structure. This was already known for o-minimal structures, but is new in the P-minimal case. Using an improved version of a result by Haskell and Macpherson (where we eliminated the assumption of definable Skolem functions, see Lemma 50), we were able to give a very short proof of this result.

The first main result is a p-adic analogue of the result we mentioned earlier for o-minimal functions.

Theorem 46. *Let $f : X \subset K \to K$ be a function definable in a strictly P-minimal structure. Then f is strictly differentiable on a cofinite subset of X.*

It will then be straightforward to show the second main result of this chapter:

Theorem 47 (Local Jacobian Property). *Let $f : X \subset K \to K$ be a function definable in a strictly P-minimal structure. There exists a finite set $I \subset X$, and a finite partition of $X \setminus I$ into definable open sets X_i, such that either $f|_{X_i}$ is constant on X_i, or the following holds on X_i: for every x in X_i, there is an open ball $B \subset X_i$ containing x, such that the map $f|_B$ satisfies the following properties:*

(a) $f|_B$ is a bijection, and $f(B)$ is a ball,

(b) f is strictly differentiable on B with strict derivative Df,

(c) $|Df|$ is constant on B,

(d) for all $x, y \in B$, one has that $|Df||x-y| = |f(x) - f(y)|$.

A global version of the above result was originally proven for semi-algebraic and subanalytic sets by Cluckers and Lipshitz [10]. Among other applications, it can be used in the study of p-adic and motivic integrals, see e.g. [13]. The (global) Jacobian Property is also a valuable tool in the study of the geometry of definable sets (see e.g. [9] or [8], where Lipschitz continuity was investigated). It is still an open question whether a global version of the Jacobian Property holds for general P-minimal structures.

Let us now return to the start of the introduction. The conjecture stated there is an immediate consequence of the Local Jacobian Property. If we combine this with Lemma 42, we obtain:

Theorem 48 (p-adic Local Monotonicity). *Let $f : X \subset K \to K$ be a function definable in a strictly P-minimal structure $(K, \mathcal{L})$. There exist definable disjoint subsets U, V of X, with $X \setminus (U \cup V)$ finite, such that*

(a) f is continuous on $U \cup V$,

(b) there exists a finite partition of U into sets U_i, such that $f|_{U_i}$ is constant,

(c) $f|_V$ *is locally strictly monotone.*

In section 4.2.2, we show that our main results hold for p-adic fields (i.e. finite field extensions of $\mathbb{Q}_p$). As a next step, we generalize to definable families of functions in section 4.2.3. This will allow us to deduce the validity of our results for the wider class of strictly P-minimal structures.

4.2 Proofs of the main results

We start with some observations on P-minimal functions. First, it is easy to see that the following lemma, which was originally proven by Denef [15, Lemma 7.1] for semi-algebraic sets, is in fact valid for P-minimal structures in general.

Lemma 49 (Denef)**.** *Let $S \subset K^{m+q}$ be a set definable in a P-minimal structure $(K, \mathcal{L})$. Let $\pi_m : K^{m+q} \to K^m$ denote the projection onto the first m coordinates.*

Assume there exists $M \geq 1$ such that for all $y \in \pi_m(S)$, the fibers $\pi_m^{-1}(y)$ are nonempty and contain at most M points. Then there exists a definable function $g : \pi_m(S) \to S$, such that $(\pi_m \circ g)(y) = y$ for all $y \in \pi_m(S)$.

One of the questions posed by Haskell and Macpherson in [23] was whether the assumption of definable Skolem functions could be eliminated from Remark 5.5 of their paper. Since they only needed Skolem functions for finite fibers of the same size, the result from Lemma 49 suffices. Therefore we have that:

Lemma 50. *Let $f : X \subset K^n \to K$ be a function definable in a P-minimal structure $(K, \mathcal{L})$. Let Y be the set*

$$Y = \{y \in X \mid f \text{ is defined and continuous in a neighbourhood of } y\},$$

then $\dim(X \setminus Y) < \dim(X)$*.*

Recall that the dimension of a definable set $X \subset K^n$ is the greatest integer k for which there exists a projection map $\pi : K^n \to K^k$, such that $\pi(X)$ has non-empty interior in K^k (we refer to [23] for more details).

We will also need the fact that the finite fibers of a definable function $f : K \to K$ are uniformly bounded:

Lemma 51. *Let $f : K \to K$ be a P-minimal function. There exists an integer M_f, such that if the fiber $f^{-1}(y)$ is finite for some $y \in f(K)$, then it contains at most M_f elements.*

Proof. This follows immediately from Lemma 5.3 of [23]. $\square$

4.2.1 Preliminary lemmas and definitions

Let us first show that if K is a p-adic field, then a P-minimal definable function $f : K \to K$ with zero derivative must be piecewise constant.

Proof of Lemma 44. By P-minimality, the domain of f is a finite union of points and open sets, so we may as well assume that $\mathrm{dom}(f)$ is an open set U. Fix $\epsilon > 0$. For every $x_0 \in U$, the fact that $f'(x_0) = 0$ implies that there exists $\delta_{x_0} > 0$, such that for all t with $|t| < \delta_{x_0}$,

$$|f(x_0 + t) - f(x_0)| < \epsilon|t| < \epsilon\delta_{x_0}. \tag{4.2}$$

Note that we may assume that $\delta_{x_0} \in |K|$. Since every open set in K can be covered by a countable number of disjoint balls, we can write $U = \bigcup_{i=1}^{\infty} B(x_i, \delta_{x_i})$. Formula (4.2) implies that $f(B(x_i, \delta_{x_i}))$ is contained in a ball with radius $\epsilon\delta_{x_i}$. Let μ be the Haar measure on K, normalized such that $\mu(\mathcal{O}_K) = 1$. Clearly, $\mu(B(x, \delta)) = \delta$ if $\delta \in |K|$. Now estimate the volume of $f(U)$:

$$\mu(f(U)) \leq \sum_{i=1}^{\infty} \mu(f(B(x_i, \delta_{x_i}))) \leq \sum_{i=1}^{\infty} \epsilon\delta_{x_i} = \epsilon\mu\Big(\bigcup_{i=1}^{\infty} B(x_i, \delta_{x_i})\Big),$$

hence $\mu(f(U)) \leq \epsilon\mu(U)$. Since the choice of $\epsilon > 0$ was arbitrary, we conclude that $f(U)$ has measure zero and hence, by P-minimality, is a finite set. One can then partition the domain into a finite union of points and open sets, on each of which the image is constant. $\square$

Lemma 52. *Let K be a p-adic field and let $f : X \subset K \to K$ be a function definable in a P-minimal structure. There exists a finite partition of X*

in definable sets $X = \cup_i X_i$ *such that for each* i, $f\big|_{X_i}$ *is either injective or constant.*

Proof. First note that the piece of the domain on which f is locally constant is a definable set X_0, consisting of the points $x \in X$ that satisfy the formula $\phi(x)$:

$$\phi(x) \leftrightarrow (\exists y)(\exists r)(\forall z)[f(x) = y \wedge |z - x| < r \rightarrow f(z) = y].$$

Since f is locally constant on X_0, $f'(x) = 0$ on X_0. Applying Lemma 44, we can then partition X_0 into a finite number of sets, on each of which f is constant.

Now consider the set $A = X \setminus X_0$. By P-minimality, any fiber $f^{-1}(y)$ with $y \in f(A)$ will be finite. Moreover, there exists an upper bound M_f for the size of these fibers, because of Lemma 51.

We can use the following procedure to partition A into a finite number of sets X_i, such that $f\big|_{X_i}$ is injective.

Applying Lemma 49 to the graph of $f\big|_A$, we can find a definable function g_1 that chooses a point x in every fiber $f^{-1}(y)$, for $y \in f(A)$. We can then put $X_1 = \{g_1(y) \mid y \in f(A)\}$. Then $f\big|_{X_1}$ is injective by construction.

Repeating the procedure for $A \setminus X_1$, we can construct a set X_2 on which f is injective, and so on. Lemma 51 ensures this algorithm will stop after at most M_f steps, so that we indeed obtain a finite partition. $\square$

Lemma 53. *Let* K *be a* p*-adic field and* $f : X \subset K \to K$ *a* P*-minimal function. There exists a finite subset* $I \subset X$ *such that for every* x_0 *in* $X \setminus I$, *with* $f(x_0) = y_0$, *the following holds:*

If $|f'(x_0)| = +\infty$, *then* f *is locally injective around* x_0 *and* $(f^{-1})'(y_0) = 0$.

Proof. By Lemma 52, one can partition X into a finite number of sets Y_i, such that $f\big|_{Y_i}$ is either injective or constant. Note that one only needs to consider those sets Y_i on which f is injective, since $f' = 0$ if f is constant. By Lemma 42 and P-minimality there exists a finite set I such that, if we put $X_i = Y_i \setminus I$, then X_i is open, and f^{-1} is continuous on $f(X_i)$.

Let $x_0 \in X_i$ be such that $|f'(x_0)| = +\infty$. That $|f'(x_0)| = +\infty$ means that for every $M > 1$, there exists $\delta > 0$ such that for all t with $|t| < \delta$,

$$\left| \frac{f(x_0 + t) - f(x_0)}{t} \right| > M. \tag{4.3}$$

Now if $(f^{-1})'(y_0) \neq 0$, then there exist $\epsilon > 0$ and s arbitrarily close to 0 such that

$$\left| \frac{f^{-1}(y_0 + s) - f^{-1}(y_0)}{s} \right| > \epsilon. \tag{4.4}$$

Now choose $M = 1/\epsilon$ and let δ be such that (4.3) holds for M. By the continuity of f^{-1} around y_0, for s close enough to 0, $f^{-1}(y_0 + s)$ lies in $B(x_0, \delta)$. Therefore $f^{-1}(y_0 + s) = x_0 + t$ for some t with $|t| < \delta$, and hence

$$\left| \frac{s}{t} \right| = \left| \frac{y_0 + s - y_0}{t} \right| = \left| \frac{f(x_0 + t) - f(x_0)}{t} \right| > M, \tag{4.5}$$

but then (4.4) and (4.5) imply that

$$\epsilon < \left| \frac{t}{s} \right| < 1/M = \epsilon,$$

which is a contradiction. $\square$

To show that an o-minimal function f is differentiable, it suffices to check that the left and right derivative of f are equal. Unfortunately, in the p-adic case we will have to deal with more possible directions. The next proposition shows that there are only finitely many possibilities.

Proposition 54. *Let $X \subset K^n$ be a set definable in a P-minimal structure. Write $\overline{X}$ for the topological closure of X. Then* $\dim(\overline{X} \setminus X) < \dim(X)$.

Proof. Let $F : \overline{X} \to K$ be the function taking the value 1 on X and 0 outside X. Applying Lemma 50, we find that $\dim(\overline{X} \setminus \operatorname{int}(X)) < \dim(\overline{X})$. Since $\dim(X) = \dim(\overline{X})$, a straightforward computation now yields the required result. $\square$

Corollary 55. *Let $f : K \to K$ be a P-minimal function. Then for each $x_0 \in K$, the limit* $\lim_{t \to 0}(f(x_0 + t) - f(x_0))/t$ *takes only a finite number of values.*

Proof. Fix $x_0 \in K$ and consider the function

$$g : K^\times \to K : t \mapsto \frac{f(x_0 + t) - f(x_0)}{t}.$$

Since the graph of g has dimension 1, Proposition 54 implies that the set $\overline{\Gamma(g)} \setminus \Gamma(g)$ has dimension zero. This proves the corollary, since all the limit values of $\lim_{t\to 0}(f(x_0 + t) - f(x_0))/t$ lie in the projection of $\overline{\Gamma(g)} \setminus \Gamma(g)$ onto the second coordinate, which is a definable subset of K with dimension zero and hence, by P-minimality, is finite. □

Definition 56. *Fix a positive integer n and an element $\lambda \in K^\times$. Define the* directional derivative *along λ with respect to n in the point $x_0 \in K$ to be*

$$f'^{\lambda}_{(n)}(x_0) = \lim_{t\to 0,\, t\in\lambda P_n^\times} \frac{f(x_0 + t) - f(x_0)}{t}, \tag{4.6}$$

if this limit exists. If n is clear from the context we will omit the index n and just write $f'^{\lambda}(x_0)$.

The next lemma and its corollary explain why it suffices to consider these directional derivatives.

Lemma 57. *Let $(K, \mathcal{L})$ be a P-minimal structure, K a p-adic field, and let $f : K \to K$ be a P-minimal function. For every $x_0 \in K$, there exists $n \in \mathbb{N}$, such that for all $\lambda \in K^\times$, either the limit $f'^{\lambda}_{(n)}(x_0)$ exists, or $|f'^{\lambda}_{(n)}(x_0)| = +\infty$.*

Moreover, given any sequence (t_j) with $\lim t_j \to 0$, for which the limit $L = \lim_{j\to\infty} \frac{f(x_0+t_j)-f(x_0)}{t}$ exists, there exists $n \in \mathbb{N}$ and $\lambda \in K^\times$ such that this limit L equals $f'^{\lambda}_{(n)}(x_0)$.

Proof. Fix x_0, and let g be the quotient function $g(t) = \frac{f(x_0+t)-f(x_0)}{t}$. By Corollary 55, there exist only finitely many values y_i for which there is a sequence $(t_j^{(i)})$ such that $g(t_j^{(i)}) \to y_i$ if $t_j^{(i)} \to 0$. Choose disjoint balls B_i, each containing exactly one of the limit points y_i. Let B be a ball with center 0. Now put $D_i = g(B) \cap B_i$, and $D = g(B) \setminus \cup_i B_i$, so that the sets $g^{-1}(D)$ and $g^{-1}(D_i)$ form a finite partition of B into definable sets.

Clearly, if the sequence $g(t_j^{(i)})$ tends to y_i, then (the tail of) the sequence $(t_j^{(i)})$ is contained in $g^{-1}(D_i)$. Similarly, the only sequences contained in $g^{-1}(D)$ are those for which $|g(t_j)| \to +\infty$. To see this, consider a sequence (t_j) with $t_j \to 0$, contained in $g^{-1}(D)$, and assume that $|g(t_j)|$ is bounded for all j. Then the set $G = \{g(t_j) \mid j \in \mathbb{N}\}$ is a bounded set, which can be assumed to be infinite. Our assumptions on K imply that the valuation ring $\mathcal{O}_K$ is compact, and hence the closure $\overline{G}$ must be compact, since for some $m \in \mathbb{Z}$, it is a closed subset of the compact set $\pi^m\mathcal{O}_K$. Therefore, $\overline{G}$ must contain a limit point, which must necessarily be one of the points y_i. Since $G \cap (\cup_i B_i) = \emptyset$ by construction, we obtain a contradiction.

Each set $g^{-1}(D)$ or $g^{-1}(D_i)$ can be partitioned in cells C (in the sense of [14], [15], and [16], using the predicates P_n instead of $Q_{m,n}$). In this way we also get a cell decomposition of B. It is easy to check that if $0 \in \overline{C}$, and if we choose $\gamma \in \Gamma_K$ big enough, then for some $n \in \mathbb{N}$ and $\lambda \in \Lambda_n$,

$$C \cap \{\mathrm{ord}(x) > \gamma\} = \lambda P_n^{\times} \cap \{\mathrm{ord}(x) > \gamma\},$$

implying that C and $\lambda P_n^{\times}$ contain the same sequences converging to 0. (Note that we can use the same value of n in all cells). Since the sets $g(C)$, by construction, contain at most one of the points y_i, the limits ${f'}_{(n)}^{\lambda}(x_0)$ must either be well defined, or $|{f'}_{(n)}^{\lambda}(x_0)| = +\infty$. $\square$

Corollary 58. *Let $(K, \mathcal{L})$ be a P-minimal structure, K a p-adic field, and let $f : K \to K$ be a P-minimal function. If for some $x_0 \in K$, the derivative $f'(x_0)$ does not exist, then either there are λ, n such that $|f'^{\lambda}_{(n)}(x_0)| = +\infty$, or, if all directional derivatives are bounded, there exist n, λ, μ such that $f'^{\lambda}_{(n)}(x_0) \neq f'^{\mu}_{(n)}(x_0)$.*

4.2.2 Proofs of the main results (for p-adic fields)

Throughout this section we will assume that we work in a P-minimal structure $(K, \mathcal{L})$ and that K is a p-adic field. Also, f will always denote a P-minimal function. The main step in the proof of Theorem 46 will be to show that sets of the following type are finite.

Definition 59. *For every positive integer n we define*

$$S_n = \left\{ x_0 \in K \;\middle|\; \begin{array}{l} \text{the limit } {f'}^{\lambda}_{(n)}(x_0) \text{ exists for all } \lambda \text{ in } \Lambda_n, \text{ and} \\ \text{there exist } \lambda, \mu \in \Lambda_n \text{ such that } {f'}^{\lambda}_{(n)}(x_0) \neq {f'}^{\mu}_{(n)}(x_0) \end{array} \right\}$$

and

$$T_n = \{x_0 \in K \mid \text{ there exists } \lambda \in \Lambda_n \text{ such that } |{f'}^{\lambda}_{(n)}(x_0)| = +\infty\}.$$

In order to prove Theorem 46, it will be sufficient to show that both $\cup_n S_n$ and $\cup_n T_n$ are finite, because of Corollary 58.

Lemma 60. *The set S_n is finite for every $n > 0$.*

Proof. Assume that S_n is infinite for some $n > 0$. By P-minimality it must then contain a ball B. By Lemma 42, after shrinking B if necessary, we may assume that for every $\lambda \in \Lambda_n$, ${f'}^{\lambda}$ is continuous on B.

Fix $x_0 \in B$. By the definition of S_n, there exist $\lambda, \mu \in \Lambda_n$ such that ${f'}^{\lambda}(x_0) \neq {f'}^{\mu}(x_0)$. After replacing f by $f(x) - {f'}^{\lambda}(x_0) \cdot x$ and rescaling, we may assume that ${f'}^{\lambda}(x_0) = 0$ and ${f'}^{\mu}(x_0) = 1$. By Hensel's Lemma, there exists m such that $1 + \pi^m \mathcal{O}_K \subset P_n^{\times}$. Fix $0 < \epsilon < |\pi^m|$. Because ${f'}^{\mu}$ is continuous, the following conditions hold if we choose $t_\lambda \in \lambda P_n^{\times}$ and $t_\mu \in \mu P_n^{\times}$ to be small enough:

$$|f(x_0 + t_\lambda) - f(x_0)| < \epsilon |t_\lambda|, \tag{4.7}$$

$$|f(x_0 + t_\mu) - f(x_0)| = |{f'}^{\mu}(x_0)||t_\mu|, \tag{4.8}$$

$$|{f'}^{\mu}(x_0 + t_\lambda)| = |{f'}^{\mu}(x_0)| = 1. \tag{4.9}$$

By changing our choices for t_λ and t_μ (choosing a smaller ϵ if necessary) we can moreover assume that $|t_\mu| = \epsilon |t_\lambda|$. By our choice of m and ϵ, we have that $t_\lambda + t_\mu = t_\lambda(1 + \frac{t_\mu}{t_\lambda}) \in \lambda P_n^{\times}(1 + \pi^m \mathcal{O}_K) \subset \lambda P_n^{\times}$. By (4.9), equation (4.8) also holds for x_0 replaced by $x_0 + t_\lambda$, so that

$$|f(x_0 + t_\lambda + t_\mu) - f(x_0 + t_\lambda)| = |{f'}^{\mu}(x_0 + t_\lambda)||t_\mu| = |t_\mu|. \tag{4.10}$$

On the other hand, since $t_\lambda + t_\mu \in \lambda P_n^\times$, t_λ can be replaced by $t_\lambda + t_\mu$ in (4.7), so

$$|f(x_0 + t_\lambda + t_\mu) - f(x_0)| < \epsilon |t_\lambda + t_\mu| = \epsilon |t_\lambda|.$$

But then $|f(x_0 + t_\lambda + t_\mu) - f(x_0 + t_\lambda)|$ is equal to

$$|(f(x_0 + t_\lambda + t_\mu) - f(x_0)) - (f(x_0 + t_\lambda) - f(x_0))| < \epsilon |t_\lambda| = |t_\mu|.$$

This contradicts (4.10), which finishes the proof. □

Corollary 61. *The set $\cup_n S_n$ is finite.*

Proof. By Lemma 60, $\cup_n S_n$ is countable. It therefore suffices to show that $\cup_n S_n$ is definable, because in a P-minimal structure every countable, definable subset of K is finite. The following formula $\psi(x)$ expresses that all the directional derivatives are bounded:

$$\psi(x) \leftrightarrow (\exists t_1, t_2)(\forall z)\left[0 < |z| < |t_1| \to \left|\frac{f(x+z) - f(x)}{z}\right| < |t_2|\right].$$

The formula $\phi(x)$ expresses that $f'(x)$ does not exist:

$$\phi(x) \leftrightarrow \neg(\exists L)(\forall t_1)(\exists t_2)(\forall z)\left[0 < |z| < |t_2| \to \left|\tfrac{f(x+z)-f(x)}{z} - L\right| < |t_1|\right].$$

Hence $\cup_n S_n$ is defined by the formula $\psi(x) \wedge \phi(x)$, because of Corollary 58. □

Lemma 62. *The set T_n is finite for every $n > 0$.*

Proof. We write $T_n = T_n^0 \cup T_n^\infty$ where

$$T_n^0 = \{x_0 \in K \mid \exists \lambda, \mu \in \Lambda_n : |{f'}^\lambda_{(n)}(x_0)| = +\infty \text{ and } |{f'}^\mu_{(n)}(x_0)| < +\infty\}$$

and

$$T_n^\infty = \{x_0 \in K \mid \forall \lambda \in \Lambda_n : |{f'}^\lambda_{(n)}(x_0)| = +\infty\}.$$

Fix $n > 0$. To simplify notation, we will omit the index n and just write ${f'}^\lambda$. The proof of the finiteness of T_n^0 is very similar to the proof of the corresponding result for S_n. Therefore we only indicate the differences. After rescaling f we may assume that $|f'^{\lambda}(x_0)| = +\infty$ and $f'^{\mu}(x_0) = 1$. Formula (4.7) should be replaced by $|f(x_0 + t_\lambda) - f(x_0)| > M|t_\lambda|$, for a fixed $M > |\pi^{-m}|$, where m is as before. The remainder of the proof is left as an exercise.

Now suppose T_n^{∞} were infinite. By P-minimality, this set must contain a ball B on which $|f'| = +\infty$. By Lemma 53, we may assume that f is injective on B and that $(f^{-1}|_{f(B)})' = 0$, after shrinking B if necessary. Lemma 44 then implies that $f^{-1}|_{f(B)}$ is piecewise constant, which is clearly impossible. □

Corollary 63. *The set $\cup_n T_n$ is finite.*

Proof. As in the proof of Corollary 61, we need to verify that $\cup_n T_n$ is definable. But this is clearly the case, since one can use the formula $\neg\psi(x)$, where $\psi(x)$ is as in the proof of Lemma 61. □

Proof of Theorem 46 (for p-adic fields). By Corollary 61, Corollary 63 and the discussion right after Definition 59, we know that there is a cofinite, definable set $A \subset K$ on which f is differentiable. Since A is definable, it is a finite union of points and open sets, namely $A = \bigcup_{i=1}^{n} A_i \cup \bigcup_{i=1}^{k} \{a_i\}$.

Applying Theorem 43 yields that the definable set

$$A_i' = \{x \in A_i \mid f \text{ is strictly differentiable at } x\}$$

is dense in A_i, for $i = 1, \ldots, n$. But then the sets $I_i = A_i \setminus A_i'$ cannot contain any balls, and hence they are finite by P-minimality. So $A \setminus \bigcup_{i=1}^{n} I_i$ is a cofinite set on which f is strictly differentiable. □

We are now ready to give a proof of Theorem 47 for p-adic fields. For the sake of clarity we will restate the theorem.

Theorem (Local Jacobian Property)**.** *Let K be a p-adic field and $f : X \subset K \to K$ a P-minimal function. There exists a finite set $I \subset X$, and a finite partition of $X \setminus I$ into definable open sets X_i, such that either $f|_{X_i}$ is constant on X_i, or the following holds on X_i: for every x in*

X_i, there is an open ball $B \subset X_i$ containing x, such that the map $f|_B$ satisfies the following properties:

(a) $f|_B$ is a bijection, and $f(B)$ is a ball,

(b) f is strictly differentiable on B with strict derivative Df,

(c) $|Df|$ is constant on B,

(d) for all $x, y \in B$, one has that $|Df||x-y| = |f(x)-f(y)|$.

Proof. By Theorem 46, there exists a finite set $I \subset X$ such that f is strictly differentiable on $X \setminus I$. This proves (b). Put $X \setminus I = A_0 \cup A$, with $A_0 = \{x \in X \setminus I \mid Df(x) = 0\}$, and $A = \{x \in X \setminus I \mid Df(x) \neq 0\}$. By Lemma 44, f is then piecewise constant on A_0. By Lemma 52 we can partition A in a finite number of pieces X_i, on which f is injective. Moreover, by P-minimality, one can assume that each X_i is open (after excluding a finite number of points if necessary).

It remains to check that $(a),(c),(d)$ hold for all points of X_i. Part (c) and (d) are immediate consequences of Theorem 46: pick any $a \in X_i$. There exists a ball $B \subset X_i$ such that for all $x, y \in B$ with $x \neq y$, it holds that
$$\left|\frac{f(x)-f(y)}{x-y}\right| = |Df(a)|.$$
Consequently, we must have that $|Df(a)| = |Df(a')|$ for all $a' \in B$ (since B contains a neighborhood of a'), from which (c) and (d) follow.
That (a) holds can be seen as follows (this part of the proof is inspired by Lemma 27.4 of [42]). Fix any $a \in X_i$, and take a ball $B(a,r)$ which is small enough to assure that for all $x, y \in B(a,r)$,
$$\sup\left\{\left|\frac{f(x)-f(y)}{x-y} - Df(a)\right| : x, y \in B, x \neq y\right\} < |Df(a)|.$$
Clearly this implies that $f(B(a,r)) \subset B(f(a), |Df(a)|r)$. It suffices to check that $f|_{B(a,r)}$ is surjective. Choose $c \in B(f(a), |Df(a)|r)$. We will show that the map $x \mapsto f(x) - c$ has a zero in $B(a,r)$. For $x \in B(a,r)$, put $g(x) = x - (f(x)-c)/Df(a)$. Then g maps $B(a,r)$ into $B(a,r)$.

Moreover, for all $x, y \in B(a,r)$, we have that

$$\begin{aligned}|g(x)-g(y)| &= \left|x-y-\frac{f(x)-f(y)}{Df(a)}\right| \\ &= \left|\frac{x-y}{Df(a)}\right|\left|\frac{f(x)-f(y)}{x-y}-Df(a)\right| \\ &\leq \tau|x-y|,\end{aligned}$$

for some $0<\tau<1$. Since $g : B(a,r) \to B(a,r)$ is a contraction, the Banach fixed-point theorem yields that $B(a,r)$ contains a point z for which $g(z)=z$, and hence $f(z)=c$. □

It is then easy to deduce that Local Monotonicity (as stated in Theorem 48) holds for p-adic fields.

We can use the techniques from the proof of the Local Jacobian Property to obtain a generalisation of Lemma 44. This is probably not new, but since we could not find any reference, we give a proof in full detail. Let μ be the Haar measure on K, normalized such that $\mu(\mathcal{O}_K)=1$. We use the notation $\mu(A)=\int_A |\mathrm{d}x|$ for a measurable set $A \subset K$.

Proposition 64. *Let $(K,\mathcal{L})$ be a P-minimal structure, K a p-adic field. Let $X, Y \subset K$ be definable, measurable sets, and let $f : X \to Y$ be a definable bijection that is strictly differentiable. Then*

$$\mu(Y)=\int_X |Df(x)||\mathrm{d}x|,$$

where the equation holds in $\mathbb{R}\cup\{+\infty\}$.

Proof. Partition $X = X_0 \cup X_1$, where $X_0=\{x\in X \mid Df(x)=0\}$ and $X_1=\{x\in X \mid Df(x)\neq 0\}$. We proved in Lemma 44 that $\mu(f(X_0))=0$, so we may just as well assume that Df is nonzero on all of X. Also, we can assume that X is open, after excluding a finite number of points if necessary. Since K is a p-adic field, X can be partitioned into a countable union of disjoint balls B_i, such that $|Df|=|c_i|$ is constant on B_i and such that $|f(x)-f(y)|=|c_i||x-y|$ for all $x,y\in B_i$. As in the proof of the Local Jacobian Property, we can argue that f maps B_i bijectively onto a ball B_i' and it can be seen easily that $\mu(f(B_i)) = |c_i|\mu(B_i)$. Since the integrand takes non-negative values, we can use sigma-additivity to compute

$$\int_X |Df(x)||\mathrm{d}x| = \sum_i \int_{B_i} |Df(x)||\mathrm{d}x| = \sum_i |c_i|\mu(B_i) = \mu(Y),$$

which proves the formula. □

4.2.3 Generalisation to strictly P-minimal structures

Given a definable function $f : A \times K \subset K^{n+1} \to K$, we write $\{f_\alpha\}_{\alpha \in A}$ for the family of functions whose members are defined by putting $f_\alpha(x) = f(\alpha, x)$. Our results can be generalized to this setting.

For a set $S \subset K^{n+1}$, we let S_α denote the fiber $S_\alpha = \{x \in K \mid (\alpha, x) \in S\}$.

Theorem 65 (Strict differentiation for definable families)**.** *Let K be a p-adic field and $f : A \times K \subset K^{n+1} \to K$ a P-minimal function. There exists a definable set $S \subset A \times K$ such that for each $\alpha \in A$, S_α is a cofinite subset of K and f_α is strictly differentiable on S_α.*

Proof. The strict derivative $Df_\alpha(a)$ of f_α in a point a can be considered as the partial (strict) derivative

$$\lim_{(x,y) \to (a,a)} \frac{f(\alpha, x) - f(\alpha, y)}{x - y}$$

of f with respect to the last variable. Let S be the set consisting of points $(\alpha, a) \in A \times K$ such that f_α is strictly differentiable in a. It is easy to see that this is a definable set (the definition is similar to the formula ϕ given in the proof of Corollary 61). The fact that for each $\alpha \in A$, S_α is a cofinite set, is a direct application of Theorem 46. □

Next, we present a uniform version of the Local Monotonicity Theorem for p-adic fields.

Theorem 66 (p-adic Local Monotonicity in definable families)**.** *Let K be a p-adic field and $f : A \times K \subset K^{n+1} \to K$ a P-minimal function. Then there exist definable disjoint subsets U, V of $A \times K$ such that for each $\alpha \in A$ the following conditions hold:*

1. *$K \setminus (U_\alpha \cup V_\alpha)$ is finite,*

2. *f_α is continuous on $U_\alpha \cup V_\alpha$,*

3. *f_α is piecewise constant on U_α. More specifically, there exists a finite partition of U in definable sets U_i, such that for each $\alpha \in A$, the function f_α is constant on each of the fibers $(U_i)_\alpha$,*

4. *f_α is locally strictly monotone on V_α.*

Proof. By Theorem 65, there exists a definable subset $S \subset A \times K$ such that for each $\alpha \in A$, S_α is a cofinite set on which f_α is strictly differentiable. Let U be the set of all points $(\alpha, x) \in S$ such that $Df_\alpha(x) = 0$, and put $V = S \setminus U$. This proves (1) and (2). Part (4) holds as a direct consequence of the Local Jacobian Property applied to $f_\alpha\big|_{V_\alpha}$.

By Lemma 44, there exists a finite partition of U_α, such that f_α is constant on each part. We will now show that this partition can be taken uniformly in the parameter α.

Let $S_{\mathrm{Im},\alpha} = \{y \in K \mid \exists x : (\alpha, x) \in U \text{ and } y = f_\alpha(x)\}$. For each $\alpha \in \pi_n(U)$, where π_n denotes the projection on the first n coordinates, this set is finite by Lemma 44. Applying Lemma 5.3 of [23] yields that there exists a partition of $\pi_n(U)$ into sets $A_1, \ldots, A_k$, and an integer M such that for $\alpha \in A_i$, the set $S_{\mathrm{Im},\alpha}$ contains at most M elements. Now Lemma 49 asserts that there is a definable way to choose an element $y_0(\alpha)$ from each $S_{\mathrm{Im},\alpha}$. So the fibers $f^{-1}(y_0(\alpha))$ (on which f_α is constant) are uniformly definable. Repeat the process for the sets $S_{\mathrm{Im},\alpha} \setminus \{y_0(\alpha)\}$, and so on. The algorithm stops after at most M steps. This concludes the proof of (3). □

The above generalizations to families of definable functions imply that the Local Monotonicity Theorem is valid for any strictly P-minimal structure.

Proof of Theorem 48. Let $(K, \mathcal{L})$ be a strictly P-minimal structure, and $f : K \to K$ an $\mathcal{L}$-definable function.

If the definition of f contains field parameters, one can replace these by variables α, and consider f to be a member of a family $\{g_\alpha\}_{\alpha \in K^n}$, which is defined by a parameter-free formula $\psi(\alpha, x, y)$. This formula ψ can then be interpreted in any $\mathcal{L}$-structure.

By our assumption, there exists a finite extension K' of $\mathbb{Q}_p$ which has the same $\mathcal{L}$-theory as K. We have already shown that the Local Monotonicity Theorem is valid for families of functions over K', so one only needs to check that there exists an $\mathcal{L}$-sentence asserting this fact. (As K and K' have the same $\mathcal{L}$-theory, this will imply that the theorem also holds for the original family $\{g_\alpha\}_{\alpha \in K^n}$. Since f is a member of this family, this proves that the Local Monotonicity Theorem holds for f.)

It is clear that parts (2), (3) and (4) of Theorem 66 can be expressed using a first order-formula. For part (1), one can use the fact that in a P-minimal structure, a definable set is cofinite if and only if its complement does not contain a ball. This clearly is a first-order condition. $\square$

By the same reasoning, the proofs of Theorem 46 and Theorem 47 can also be generalized to strictly P-minimal structures. Therefore, Theorem 65 and Theorem 66 also hold for strictly P-minimal structures.

Chapter 5

Discussion: towards future research

Often in Mathematics, solving one problem gives rise to more than one new problem. Therefore, mathematical research is only rarely a *good news only* story. Every assumption in every theorem is probably due to the fact that without that assumption, something goes wrong. And however a mathematician is never proud of the *bad news* side of the story, we believe it is rather important that it is told anyway. For the sake of future researchers, for example, so they know where things can go wrong and to what perspective results can be ameliorated.

In this rather short chapter, we discuss the main theorems of the previous two chapters and point out clearly which assumptions are needed to prevent which problem, and where we believe that the results can be generalized. We even generalize some results already ourselves, showing that on the *bad news* side, there is sometimes also good news.

5.1 Discussion on Lipschitz extensions

We use quite some advanced techniques to prove the main result of chapter 3, such as Theorem 26, which is a refined form of p-adic cell decomposition. To use these techniques, we must restrict to p-adic semi-algebraic and subanalytic structures. However, we don't use all the aspects of Theorem 26 in the construction of definable Lipschitz extensions. To name one thing, we never use the fact that the diameters of the balls of a cell above some $y \in Y$ form an arithmetic progression. It is therefore natural to ask:

***Question* 1.** Do the main results from chapter 3 still hold when considering other structures than the semi-algebraic or subanalytic structures on K, for example P-minimal structures?

Trying to answer this question, we found a generalization of the main result of chapter 3. Remarkably, this new result *simplifies* the construction of definable Lipschitz extensions to some extent.

Let K be a p-adic field and $\mathcal{L}$ a language extending $\mathcal{L}_{\text{Mac}}$. We introduce a very general notion of *cells* in the structure $(K, \mathcal{L})$. Recall the notation

$$B_{l,c,m,\xi} = \{x \in K \mid \operatorname{ord}(x - c) = l, \overline{\operatorname{ac}}_m(x - c) = \overline{\operatorname{ac}}_m(\xi)\}$$

for a ball in K, where l is an integer, m is a positive integer, $c \in K$ and $\xi \in K^\times$. Note that for the same reasons as before, the set $B_{l,c,m,\xi}$ is a definable set, where from now on by *definable* we mean definable in the structure $(K, \mathcal{L})$.

Definition 67. *Let Y be a definable set. A 0-cell over Y is the graph of a definable function $c : Y \to K$. A 1-cell over Y is a nonempty set $C \subset K \times Y$, such that for each $y \in Y' = \{y \in Y \mid (\exists x)((x, y) \in C)\}$ one has:*

$$\{x \in K \mid (x, y) \in C\} = \bigcup_{l \in L_y} B_{l,c(y),m,\xi},$$

where for every $y \in Y'$, L_y is a nonempty subset of $\mathbb{Z}$, m is a positive integer, $\xi \in K^\times$, and $c : Y' \to K$ is a definable function. For a fixed $y \in Y'$, we call the collection of balls $\cup_{l \in L_y} B_{l,c(y),m,\xi}$ the balls of the cell *above y. Say a 1-cell is* dense *if for every $y \in Y'$, the set L_y is not bounded from above. Say a 1-cell is* sparse *if for every $y \in Y'$, the set L_y is bounded from above. The set Y' is called the* base *of the cell C, c is called the* center *of the cell C, and m is called the* depth *of the cell C.*

Remark 68. The center of a dense 1-cell is unique, while the center of a sparse 1-cell is not. In a dense 1-cell C, for every y in the base of C, there are *arbitrarily small balls* of C above y. In a sparse 1-cell C, for every y in the base of C, there is a *smallest ball* of C above y.

Definition 69. *Let K be a p-adic field and $\mathcal{L}$ a language extending $\mathcal{L}_{Mac}$. Say $(K, \mathcal{L})$ is a* cell structure *if $(K, \mathcal{L})$ admits definable selection and if for every definable set Y, every definable set $X \subset K \times Y$ can be partitioned in a finite number of disjoint cells over Y, such that each cell in this partition is either a 0-cell over Y, a dense 1-cell over Y, or a sparse 1-cell over Y.*

Examples of cell structures are p-adic semi-algebraic and p-adic subanalytic structures. Indeed, for this we first note that by Proposition-Definition 14, p-adic cells are in particular *cells* in our new framework. Let $X \subset K \times Y$ be a definable set and let C be a 1-cell over Y in the p-adic cell decomposition of X of the form

$$C = \{(x, y) \in K \times Y \mid y \in Y',\ |\alpha(y)| \,\square_1\, |x - c(y)| \,\square_2\, |\beta(y)|,\ x - c(y) \in \xi Q_{m,n}\}.$$

If $\square_1$ is "no condition", then clearly C is a dense 1-cell. If $\square_1$ is "$<$", then since $\alpha : Y' \to K^\times$ is never zero, C is a sparse 1-cell.

More generally, every P-minimal structure endowed with definable selection is a cell structure. This is due to Mourgues in [34], where she proves that a P-minimal structure admits cell decomposition if and only if it has definable selection. She uses a slightly different form of p-adic cells (she uses the predicates P_n instead of $Q_{m,n}$), but up to a finite partition, the description of cells in terms of *balls of a cell* as in Proposition-Definition 14, remains valid (this relates to Lemma's 11 and 13). This is a potentially more general context, since the existence of the Jacobian Property is not known in these structures.

We can now present a generalization of the results in chapter 3 to cell structures, and in particular to P-minimal structures endowed with definable selection.

Theorem 70. *Let $(K, \mathcal{L})$ be a cell structure. Let $Y \subset K^r$ and $X \subset K$ be definable sets and let $S = X \times Y$. Let $f : S \to K^s$ be a definable function that is λ-Lipschitz in the first variable. Then f extends to a definable function $\tilde{f} : K \times Y \to K^s$ that is λ-Lipschitz in the first variable, i.e. $\tilde{f}_y$ is λ-Lipschitz for every $y \in Y$.*

Proof. We use the same techniques as in the proof of Theorem 29 to make the following simplifications: by Remark 31, we may assume that $\lambda = 1$ and $s = 1$. By Theorem 26 and (the remark after) Lemma 32 (this is the Gluing Lemma), we may assume that S is either a 0-cell, a dense 1-cell or a sparse 1-cell over Y. Furthermore, we may assume that the base of S is Y.

If S is a 0-cell with center c, extend f in the obvious way

$$\tilde{f} : K \times Y : (x, y) \mapsto f(c(y), y).$$

If S is a dense 1-cell with center c, then for every y in Y, there are arbitrarily small balls of S above y, hence we can extend f to a definable function on the union of S and the graph of c that is 1-Lipschitz in the first variable, by a topological argument (see Lemma 22). We abuse notation and call this extension f as well. Now define

$$\hat{f} : K \times Y : (x, y) \mapsto \begin{cases} f(x, y) & \text{if } (x, y) \in S; \\ f(c(y), y) & \text{otherwise.} \end{cases}$$

Clearly, $\hat{f}$ is a definable extension of f. Fix $y \in Y$, we prove that $\hat{f}_y$ is q^m-Lipschitz, where m is the depth of the cell S. Let $(t_1, y) \in S$ and $(t_2, y) \notin S \cup \{(c(y), y)\}$. Then

$$|\hat{f}(t_1, y) - \hat{f}(t_2, y)| = |f(t_1, y) - f(c(y), y)| \leq |t_1 - c(y)|. \tag{5.1}$$

There are two cases to consider:

Case 1: $|t_1 - c(y)| \neq |t_2 - c(y)|$. Then by the non-Archimedean property it follows that $|t_1 - c(y)| \leq |t_1 - t_2|$, which together with (5.1) gives $|\hat{f}(t_1, y) - \hat{f}(t_2, y)| \leq |(t_1, y) - (t_2, y)|$.

Case 2: $|t_1 - c(y)| = |t_2 - c(y)|$. Let l be such that $t_1 \in B_{l,c(y),m,\xi}$, then since $t_2 \notin B_{l,c(y),m,\xi}$, one has

$$|t_1 - t_2| > q^{-l-m} = q^{-m}|t_1 - c(y)|,$$

or equivalently

$$|t_1 - c(y)| < q^m |t_1 - t_2|.$$

Together with (5.1) we then find $|\hat{f}(t_1, y) - \hat{f}(t_2, y)| \leq q^m |(t_1, y) - (t_2, y)|$.

Now we can finish by the exact same argument as in Remark 36, defining $\tilde{f} : K \times Y \to K : (x, y) \mapsto \hat{f}(\varphi(x, y), y)$, where φ is the map *rescaling the angular component* (see Remark 36 on page 63 for all the details). It's straightforward that $\tilde{f}_y$ is 1-Lipschitz for every $y \in Y$.

If S is a sparse 1-cell with center c, for every $y \in Y$ there is a smallest ball of S above y, which we denote by B_y. The set $\{B_y \times \{y\} \mid y \in Y\}$ is clearly definable, hence by definable selection there is a function $h : Y \to K$ such that for each $y \in Y$, we have that $(h(y), y) \in \{B_y \times \{y\} \mid y \in Y\}$. Now define

$$\hat{f} : K \times Y : (x, y) \mapsto \begin{cases} f(x, y) & \text{if } (x, y) \in S; \\ f(h(y), y) & \text{otherwise.} \end{cases}$$

Again, it is clear that $\hat{f}$ is a definable extension of f. We claim that $\hat{f}$ is q^m-Lipschitz in the first variable, where m is the depth of the cell S. Fix $y \in Y$, $(t_1, y) \in S$ and $(t_2, y) \notin S$. Let us first compute

$$|\hat{f}(t_1, y) - \hat{f}(t_2, y)| = |f(t_1, y) - f(h(y), y)| \leq |t_1 - h(y)|. \tag{5.2}$$

There are again two cases to consider:

Case 1: $t_1 \in B_y$**.** Then immediately one finds $|t_1 - h(y)| < |t_1 - t_2|$, hence by (5.2) it holds that $|\hat{f}(t_1, y) - \hat{f}(t_2, y)| < |t_1 - t_2|$.

Case 2: $t_1 \notin B_y$**.** In this case, there are three calculations to be made, all based on the non-Archimedean property:

1. if $|t_2 - h(y)| < |t_1 - h(y)|$, then $|t_1 - h(y)| = |t_1 - t_2|$;
2. if $|t_2 - h(y)| = |t_1 - h(y)|$, then the exact same argument as in Case 2 when S is a dense 1-cell gives that $|t_1 - h(y)| < q^m |t_1 - t_2|$;
3. if $|t_2 - h(y)| > |t_1 - h(y)|$, then $|t_1 - h(y)| < |t_1 - t_2|$.

In all these three cases, by (5.2) we have $|\hat{f}(t_1, y) - \hat{f}(t_2, y)| \leq q^m |(t_1, y) - (t_2, y)|$.

We can finish now in the exact same way as in the case that S is a dense 1-cell, defining $\tilde{f} : K \times Y \to K : (x, y) \mapsto \hat{f}(\varphi(x, y), y)$, where φ is as above. In this way we obtain the desired definable extension of f, because $\tilde{f}$ is 1-Lipschitz in the first variable. $\square$

So far we have only constructed definable Lipschitz extensions of families of functions in one variable. This is already much better than just definable Lipschitz extensions of functions in one variable, but it is still far away from definable Lipschitz extensions in general dimension. Therefore one should definitely ask the following question:

***Question* 2.** Does the main result from chapter 3 hold in general dimension? I.e. does every definable λ-Lipschitz function $f : S \subset K^r \to K^s$ admit a definable extension $\tilde{f} : K^r \to K^s$ that is Λ-Lipschitz in *all variables*? Can Λ be taken equal to λ?

Recall that in the real case, this question has a positive answer. In the p-adic case, one could make the following observations. The Gluing Lemma (Lemma 32) allows us to restrict to the case that S is a cell over $Y \subset K^{r-1}$. Also, we may assume that $s = 1$ and $\lambda = 1$ by Remark 31. Given a definable 1-Lipschitz (in *all* variables) function $f : S \subset K \times Y \to K$, a first idea could be to take the same extension as the one we constructed in Theorem 29 and then to check whether this extension is also 1-Lipschitz in *all* variables. More concretely, the following could be a strategy to solve Question 2:

1. Try to prove a cell decomposition theorem where the center of the cell S_f in the image is 1-Lipschitz.
2. Extend the center $c' : Y \subset K^{r-1} \to K$ of the cell S_f to a 1-Lipschitz function $\tilde{c}' : K^{r-1} \to K$ using induction on the dimension.
3. Define $\tilde{f} : K^r \to K$ by sending (x, y) to either the already known extension $\tilde{f}(x, y)$ if $y \in Y$, or to $\tilde{c}'(y)$ if $y \notin Y$.
4. Prove that $\tilde{f}$ is 1-Lipschitz.

It would be remarkable if this would work, since obviously the construction from Theorem 29 is *designed* to only give Lipschitz continuity in the first variable. Still, some configurations of points do satisfy $|\tilde{f}(x_1, y_1) - \tilde{f}(x_2, y_2)| \leq |(x_1, y_1) - (x_2, y_2)|$ automatically, but problems occur when for example $(x_1, y_1) \in S$ and $(x_2, y_2) \notin S$ and moreover $x_1 = x_2$, and y_1 lies very close to y_2.

Finally, one could wonder whether the results from chapter 3 hold uniformly in p.

***Question* 3.** Is there a way to make sense of Theorem 34, Theorem 29 or Theorem 70, i.e. the results on definable Lipschitz extensions, uniformly in p?

By this we mean that the formula's defining f and $\tilde{f}$ are *uniform in K*. One approach could be to consider fields K with residue field characteristic large enough (depending on f), and to use cell decomposition results that are uniform in p from [35]. Another approach could be to fix p and to look at varying p-adic fields K, using results from [36].

5.2 Discussion on *P*-minimal structures

In chapter 4 we obtained the main results only after putting an extra condition on the P-minimal structures $(K, \mathcal{L})$ under consideration, namely that of *strict P-minimality*. This condition is nothing more than asking that there is at least one P-minimal finite field extension of $\mathbb{Q}_p$ elementary equivalent to K. We use this assumption to be able to develop the main theorems first for P-minimal finite field extensions of $\mathbb{Q}_p$, and then to use model theory to extend them to strictly P-minimal fields. Therefore one could ask the following question:

***Question* 4.** Does for every P-minimal structure $(K, \mathcal{L})$, there exist a P-minimal finite field extension of $\mathbb{Q}_p$ that is elementary equivalent to K?

If the answer to this question would be positive, the notions of *strict P-minimality* and *P-minimality* would coincide. Although this question is still wide open, there are some hints towards a negative answer: in [4] a P-minimal structure is studied that might be not strictly P-minimal. It is interesting to note that the analogue question in the real case has a negative answer. Indeed, there exist o-minimal structures that cannot live on any o-minimal expansion of the real field, see [30] and [11].

We also only obtain a *local* version of the Monotonicity Theorem in strictly P-minimal structures. A reasonable question would therefore be:

***Question* 5.** Does there exist a global Monotonicity Theorem for (strictly) P-minimal structures? I.e. does Theorem 48 hold with “locally strictly monotone” replaced by “strictly monotone”?

Here, one could take as the definition for $f : X \subset K \to K$ being “strictly monotone” the condition that for every $x, y \in X$ such that the smallest ball containing x and y is contained in X, one has:

$$|x - z| < |x - y| \Rightarrow |f(x) - f(z)| < |f(x) - f(y)|.$$

A positive answer to Question 5 would follow immediately from a global version of the Jacobian Property for strictly P-minimal fields. However, the (global) Jacobian Property is for now only known for semi-algebraic and subanalytic structures, and not for (strictly) P-minimal structures in general, see [10].

Voor de niet-wiskundige

Voor wie niet thuis is in wiskunde, maar er wel wat meer over wil te weten komen, is er dit hoofdstuk voor de niet-wiskundige. We nemen je mee in een verhaal over afstand en gaan samen op zoek naar andere afstanden dan die we gewoon zijn. De hoofdrolspelers in dit verhaal zijn de traditionele afstand, de triviale afstand en de p-adische afstand. Zet je schrap voor wat je altijd al wou weten, maar nooit durfde te vragen, over wiskunde[1].

Meten is weten: de afstand

Ons verhaal begint bij het begrip *afstand*. Afstand is van groot belang in het dagdagelijkse leven en afstanden meten doen we vaker dan we zouden denken. Hoe lang nog tot vanavond (afstand in je agenda)? Een grote of een kleine pizza (afstand van het middelpunt van je pizza tot aan de rand)? Een groot of een klein pintje (afstand van de toog tot aan de bovenkant van je glas)? Hoe ver nog naar huis (fiets, nachtbus of taxi)? Je bent steeds actief (of passief) bezig met afstanden te berekenen.

Bovendien gebruiken we allemaal *dezelfde* afstand: een meter voor jou is even lang als een meter voor hem of haar, een meter hier is evenveel als een meter daar. Toegegeven, niet iedereen zal het hebben over meters, centimeters of kilometers. Sommigen praten over duimen en voeten, over inches en mijlen of zelfs over lichtjaren en astronomische eenheden. Maar

[1]We nemen de volledige verantwoordelijkheid voor eventuele onduidelijkheden in de tekst en sporen de lezer aan niet te aarzelen om extra uitleg te vragen.

na de gepaste omrekening kunnen we allemaal begrijpen hoe lang de afstand is waarover wordt gesproken.

Dat we allemaal hetzelfde afstandsbegrip hanteren, lijkt op het eerste gezicht een logische keuze en vooral heel praktisch. Maar zou het niet leuk zijn om eens met een andere, creatievere bril naar de wereld te kijken, met een andere invulling van die eeuwenoude traditionele afstand? Dat is de volgende stap van dit verhaal: de zoektocht naar een nieuwe afstand.

Wat is dat, een afstand?

Vooraleer we op zoek kunnen gaan naar een nieuw soort afstand, moeten we natuurlijk eerst duidelijk weten hoe een afstand er uit moet zien. Met andere woorden willen we weten wat datgene is wat van iets een *afstand* maakt. Als we bijvoorbeeld de afstand meten tussen de onderkant en de bovenkant van dit blad, dan bekomen we 24cm. Een afstand is dus iets dat aan twee objecten een getal verbindt, en dat getal noemen we dan de *afstand* tussen die twee objecten[2]. Nu kan men veel mogelijke manieren bedenken om aan twee objecten een getal te verbinden, maar die komen niet allemaal in aanmerking om een afstand genoemd te worden. Wat maakt van zo'n functie een afstandsfunctie? Wat zijn met andere woorden de kenmerkende eigenschappen van een afstand? Veel wiskundigen hebben zich over deze vraag gebogen en na lang overleg zijn de volgende drie kenmerken uit de ideeënbus gekomen.

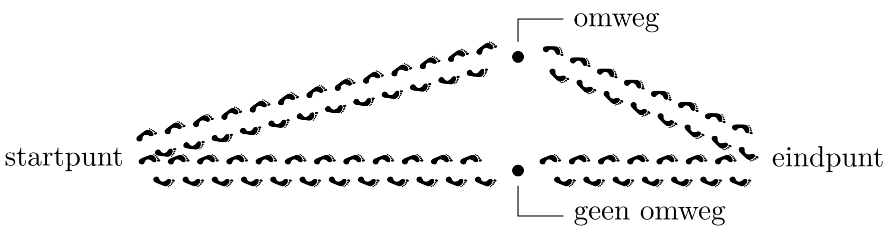
Ten eerste moet een afstandsfunctie aan elke twee objecten een *positief getal* associëren (met positief bedoelen we hier *groter dan of gelijk aan 0*). Bovendien is de afstand van een object tot zichzelf altijd *gelijk* aan 0, en de afstand tussen twee *verschillende* objecten steeds *strikt* groter dan 0. Dit is een eigenschap die we als heel natuurijk ervaren in het dagelijkse leven. Inderdaad, dat afstanden steeds positief zijn, komt neer op het feit dat als je een meter naar voor wandelt, of een meter naar achter, je in beide gevallen een afstand van 1 meter hebt afgelegd (en dus niet -1 meter in het tweede geval). Bovendien is de afstand van

[2] In de wiskunde spreekt men van een *afstandsfunctie* die aan elke twee objecten een getal associeert.

een object tot zichzelf uiteraard 0 en aangezien er geen objecten bestaan die *oneindig dicht* bij elkaar liggen, moet de afstand tussen verschillende objecten wel strikt groter zijn dan 0.

Ten tweede mag het niet uitmaken in welke volgorde we de afstand tussen twee objecten meten. De afstand tussen de onderkant en de bovenkant van dit blad is hetzelfde als de afstand tussen de bovenkant en de onderkant van dit blad. De afstand is met andere woorden *symmetrisch*.

Ten derde is er een eigenschap die iets zegt over het maken van *omwegen*, namelijk dat het altijd korter is om rechtstreeks van het ene punt naar andere te gaan, dan een omweg te maken via een derde punt. Het kan natuurlijk ook zijn dat het derde punt *op de weg ligt*, ergens tussen het startpunt en het eindpunt, en in dat geval spreken we niet van een omweg aangezien er geen extra afstand bijkomt. Met andere woorden, als het derde punt precies op de lijn tussen het startpunt en het eindpunt ligt, dan is het even lang om rechtstreeks van het startpunt naar het eindpunt te gaan, als eerst van het startpunt naar het derde punt en dan van het derde punt naar het eindpunt te gaan. Dit kenmerk van een afstandsfunctie noemen we de *driehoeksongelijkheid* en kan je visueel voorstellen als volgt:



We zetten dit nu om in een wiskundige taal. Objecten waartussen we afstanden meten, zullen we noteren met x, y en z. De afstand tussen twee objecten x en y noteren we met $d(x, y)$ (waarbij de letter "d" komt van het Engelse of Franse woord *distance*). De drie bovenstaande kenmerkende eigenschappen van een afstand kunnen we dan heel kort als volgt samenvatten: d is een afstandsfunctie als en slechts als voor alle x, y en z geldt dat

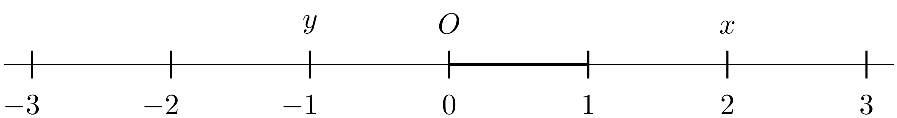
1. $d(x, y) \geq 0$ en $d(x, y) = 0 \iff x = y$;

2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq d(x, z) + d(z, y)$.

De laatste eigenschap is een samenvatting van wat hiervoor vermeld werd over het maken van omwegen: indien het punt z niet tussen x en y ligt, luidt de formule $d(x, y) < d(x, z) + d(z, y)$, indien het punt z wel tussen x en y ligt, dan krijgen we $d(x, y) = d(x, z) + d(z, y)$.

De traditionele afstand

De traditionele afstand waarmee we dagelijks in contact komen, willen we nu ook in wiskundige taal omzetten. Eerst kiezen we een gepaste voorstellingswijze voor de objecten waartussen we de afstand zullen meten. Gemakshalve bestuderen we in dit hoofdstuk enkel objecten die leven in een 1-dimensionale ruimte, of met andere woorden: punten op een rechte. We kunnen elke rechte *ijken* door één vast punt te kiezen (de *oorsprong*, aangeduid met O) en een *basiseenheid* te kiezen. Denk bijvoorbeeld aan een typische meetlat, waar het vaste punt 0 is, en de basiseenheid één centimeter is. Op die manier kan elk punt op de rechte uitgedrukt worden als een aantal keer de basiseenheid, samen met de informatie of het punt links of rechts van de oorsprong ligt.



Zo kunnen we het punt x identificeren met "twee keer de basiseenheid, rechts" en y met "één keer de basiseenheid, links". Kort duiden we x ook wel aan met "2" en y met "-1". Op die manier kunnen we aan elk punt op de (geijkte) rechte een getal associëren. De oorsprong O komt dan overeen met het getal 0.

Als we terugkijken naar de rechte hierboven, zien we duidelijk dat de afstand tussen x en y drie keer de basiseenheid is. Het is dus logisch om te zeggen dat de afstand tussen x en y gelijk is aan 3. Als x en y echter heel ver van elkaar zouden liggen, dan zou het enorm veel werk zijn om het aantal basiseenheden tussen x en y te tellen om deze afstand te bepalen. Daarom kunnen we beter naar de getallen kijken die we met x en y hebben geassocieerd. We zien namelijk dat $2 - (-1) = 2 + 1 = 3$, waar we gebruik maakten van de rekenregel $-(-\text{getal}) = +\text{getal}$. Blijkbaar kunnen we de afstand tussen twee punten dus berekenen door het getal te nemen dat hoort bij het meest rechtse punt, en daarvan het getal af te trekken dat hoort bij het meest linkse punt.

In de bovenstaande berekening lijkt het van belang te zijn dat we *eerst* het rechtse punt nemen en er *daarna* het linkse van aftrekken. Is de volgorde voor het meten van de afstand tussen twee punten dan toch van belang? Wat zou er gebeuren als we de volgorde in deze berekening zouden omkeren? Als we dit uitrekenen, krijgen we: $(-1) - 2 = -3$. Dit is echter een negatief getal en we hadden eerder al beargumenteerd dat een afstand altijd iets *positiefs* moest zijn. Gelukkig zien we wel het getal 3 verschijnen, wat de echte afstand is tussen x en y, dus kunnen we dit probleem oplossen door het minteken voor de 3 te negeren.

Hier komt het begrip *absolute waarde* op de proppen: de absolute waarde van een positief getal is gewoon dit getal zelf, de absolute waarde van een negatief getal is het getal zonder het minteken. De absolute waarde van een getal wordt aangeduid door $|\text{getal}|$. Enkele voorbeelden: $|2| = 2$, $|-3| = 3$, $|-2014| = 2014$, $|0| = 0$, etc.

Met behulp van de absolute waarde kunnen we de afstand tussen twee punten nu mooier noteren. Vanaf nu *identificeren* we een punt x op de rechte met het getal dat ermee overeenkomt na het kiezen van een oorsprong en een basiseenheid (dit getal wordt de *coördinaat* van x genoemd). Als we het voortaan hebben over een punt x bedoelen we dus dat x een getal voorstelt. De *(traditionele) afstand* $d(x, y)$ tussen twee punten x en y op de rechte kunnen we dan definiëren als de absolute waarde $|x - y|$. Met andere woorden: $d(x, y) = |x - y|$.

Om na te gaan of dit een *afstandsfunctie* is, zoals we hierboven hebben gedefinieerd, moeten we de drie kenmerkende eigenschappen nagaan. Ten eerste is $|x - y|$ groter of gelijk aan 0 wegens de manier waarop de absolute waarde is gedefinieerd. Bovendien is $|x - y|$ *gelijk* aan 0 enkel wanneer $x - y$ gelijk is aan 0 en dus x gelijk is aan y. Ten tweede geldt de symmetrie $|x - y| = |y - x|$, dit zagen we ook al in het voorbeeld hierboven. Ten derde kan men nagaan dat de driehoeksongelijkheid geldt, dit laten we als oefening voor de lezer.

De triviale en *p*-adische afstand

Hoe zit het dan met andere mogelijke afstanden? In het geval van de traditionele afstand zagen we dat het begrip *afstand* sterk verbonden is met het begrip *absolute waarde*.

Even wat meer over die absolute waarde. De absolute waarde meet in zekere zin de *grootte* van een getal. Hierbij maken we de opmerking dat een getal als -1000 niet *klein* is, het is eerder heel groot, maar wel negatief[3]. Dat komt overeen met het feit dat $|-1000| = 1000$. Met andere woorden geeft de absolute waarde een maat aan een getal, onafhankelijk of het nu negatief of positief is.

Het feit dat er verschillende soorten afstanden bestaan, komt eigenlijk doordat er verschillende soorten absolute waardes bestaan. Er zijn met andere woorden verschillende mannieren om aan een getal een grootte toe te kennen. Als we dan een andere absolute waarde nemen, dus een andere manier om aan een getal een grootte toe te kennen, en de afstand tussen x en y nog steeds definiëren als de (nieuwe) absolute waarde van $x - y$, dan zal dit aanleiding geven tot een nieuwe afstand.

Net zoals niet elke functie een afstandsfunctie is, maar aan drie (heel natuurlijke) kenmerkende eigenschappen moet voldoen, zal ook niet alles in aanmerking komen om een absolute waarde genoemd te mogen worden. We kijken eerst naar drie eigenschappen van de traditionele absolute waarde en zullen die dan later gebruiken om het begrip *absolute waarde* meer algemeen te definiëren.

Ten eerste geldt er voor alle getallen x dat $|x| \geq 0$. Bovendien weten we dat als $x \neq 0$, dan $|x| > 0$, en dat $|0| = 0$. Ten tweede kunnen we iets zeggen over de absolute waarde van het product van twee getallen. Bijvoorbeeld: $|(-3) \cdot 7| = |-21| = 21$ en $|-3| \cdot |7| = 3 \cdot 7 = 21$. Meer

[3] We zeggen ook niet dat onze schuld bij de bank *heel klein* is als we heel hard in het rood staan.

algemeen geldt voor alle getallen x en y dat $|xy| = |x||y|$. Ten derde kunnen we ons afvragen of hetzelfde ook waar is voor de som van twee getallen in plaats van het product. Met andere woorden, geldt voor alle getallen x en y dat $|x + y| = |x| + |y|$? Dit is zo als x en y beiden positief of negatief zijn. Bijvoorbeeld: $|3 + 7| = 10 = |3| + |7|$ en $|-3+(-7)| = |-10| = 10 = |-3|+|-7|$. Als x en y echter een verschillend teken hebben, dan zal steeds $|x+y| < |x|+|y|$. Bijvoorbeeld: $|-3+7| = 4$ en dit is strikt kleiner dan $|-3| + |7| = 10$. Met andere woorden geldt er steeds dat $|x + y| \leq |x| + |y|$.

We nemen deze drie eigenschappen van de traditionele absolute waarde nu als definitie voor elke andere absolute waarde. Met andere woorden is $|\ |$ een *absolute waarde* als en slechts als voor alle x en y geldt dat:

1. $|x| \geq 0$ en $|x| = 0 \iff x = 0$;
2. $|xy| = |x||y|$;
3. $|x + y| \leq |x| + |y|$.

Men kan aantonen dat deze drie voorwaarden ervoor zorgen dat als $|\ |$ een absolute waarde is, dan $d(x, y) = |x - y|$ een afstandsfuntie is. We zullen in wat volgt twee nieuwe absolute waardes definiëren, die dus elk een nieuwe afstand bepalen.

De *triviale absolute waarde* is een absolute waarde die aan elk getal de grootte 1 toekent, behalve aan het getal 0, die grootte 0 krijgt. We noteren deze absolute waarde met $|x|_{\text{triv}}$ (ga zelf na dat dit een absolute waarde is). Er geldt dus voor elk getal x verschillend van 0 dat $|x|_{\text{triv}} = 1$ en er geldt dat $|0|_{\text{triv}} = 0$.

Voor alle punten x en y definiëren we de afstand tussen x en y of de afstand $\mathrm{d}_{\text{triv}}(x, y)$ als volgt: $\mathrm{d}_{\text{triv}}(x, y) = |x - y|_{\text{triv}}$. Dit komt overeen met de volgende definitie: $\mathrm{d}_{\text{triv}}(x, y) = 1$ als $x \neq y$ en $\mathrm{d}_{\text{triv}}(x, y) = 0$ als $x = y$. Met andere woorden: de aftand tussen eender welke twee punten is 1, behalve tussen een punt en zichzelf, die afstand is 0. Dat d_{triv} een afstandsfunctie is, kan eenvoudig worden nagaan (dit volgt ook uit het feit dat $|\ |_{\mathrm{triv}}$ een absolute waarde is, zoals hierboven werd opgemerkt). Deze afstand wordt de *triviale*[4] *afstand* genoemd.

Je kan je veel filosofische vragen stellen bij deze afstand. Bijvoorbeeld hoe het zou zijn om in een wereld te leven die is uitgerust met de triviale afstand. Op het eerste gezicht lijkt dit een leuke wereld, waar veel problemen verdwijnen: iedereen woont op afstand 1 van zijn werk, dus pendelen wordt een pak minder vervelend; verre vrienden en familie zijn plots zo ver weg niet meer, ze wonen allemaal dicht in de buurt. Toch zou dit ook een enorm eenzame wereld zijn, aangezien iedereen wel dicht bij elkaar is, maar nooit *heel erg dicht*.

De triviale afstand kan een beetje *gekuntseld* lijken (zo gemaakt om aan de drie voorwaarden van een afstandsfunctie te voldoen). Laten we dan eens kijken naar een minder artificieel voorbeeld. Voor redenen die later duidelijk zullen worden, definiëren we deze nieuwe afstand eerst voor gehele getallen[5]. Laten we eerst een nieuwe absolute waarde, of nog een nieuwe *grootte* van een geheel getal definiëren. Later zullen we dan opnieuw de afstand tussen twee gehele getallen x en y definiëren als de absolute waarde van $x - y$.

Kies een priemgetal[6] en noem dat p voor de rest van dit hoofdstuk. In alle voorbeelden die volgen zullen we zelf 3 nemen als priemgetal. Neem nu een geheel getal x en schrijf dit als $x = p^k \cdot x'$, waarbij x' niet deelbaar is door p. Bijvoorbeeld als $p = 3$ en $x = 18$, dan is $18 = 3^2 \cdot 2$, dus is in dit geval $k = 2$ en $x' = 2$. Met andere woorden is k het grootste getal zodat p^k een deler is van x. Inderdaad, in ons voorbeeld zien we dat 18 deelbaar is door $3^2 = 9$, maar niet door $3^3 = 27$.

We definiëren de nieuwe absolute waarde van x dan als $\frac{1}{p^k}$. Men noemt

[4]"Triviaal" is een woord dat vaak gebruikt wordt in de wiskunde. Het is echter moeilijk de precieze betekenis van het woord te bepalen. Het wordt meestal gebruikt om aan te duiden dat een wiskundig object het eenvoudigste is in zijn soort. Het is een misvatting dat triviale objecten *onbelangrijk* zouden zijn, meestal is het precies andersom!

[5]De gehele getallen zijn de getallen $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

[6]Een *priemgetal* is een geheel getal groter of gelijk aan 2, dat enkel zichzelf en 1 heeft als positieve delers. De priemgetallen tussen 0 en 100 zijn: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 en 97. Euclides bewees dat er oneindig veel priemgetallen bestaan!

dit de *p-adische*[7] *absolute waarde* van x en noteert dit als $|x|_p$. Er geldt dus dat $|x|_p = \frac{1}{p^k}$. Voor het geheel getal 0 stellen we per definitie dat $|0|_p = 0$.

Bijvoorbeeld: de 3-adische absolute waarde van 18 is $\frac{1}{3^2} = \frac{1}{9} = 0,11\ldots$ en de 3-adische absolute waarde van 405 is

$$\frac{1}{3^4} = 0.012345679012345679\ldots,$$

want $405 = 3^4 \cdot 5$ en dus is $|405|_3 = \frac{1}{3^4}$.

Voorlopig is de p-adische absolute waarde $|\ |_p$ enkel nog maar gedefinieerd voor gehele getallen. We kunnen $|\ |_p$ nu uitbreiden tot alle *rationale getallen*[8]. Voor een rationaal getal $\frac{x}{y}$ definiëren we $|\frac{x}{y}|_p = \frac{|x|_p}{|y|_p}$. Bijvoorbeeld:

$$\left|\frac{405}{18}\right|_3 = \frac{|405|_3}{|18|_3} = \frac{1/3^4}{1/3^2} = \frac{3^2}{3^4} = \frac{1}{3^2} = 0.11\ldots.$$

Een tweede voorbeeld is

$$\left|\frac{18}{405}\right|_3 = \frac{|18|_3}{|405|_3} = \frac{1/3^2}{1/3^4} = \frac{3^4}{3^2} = 3^2 = 9.$$

Als laatste voorbeeld berekenen we dat

$$\left|\frac{4}{500}\right|_3 = \frac{|4|_3}{|500|_3} = \frac{1/3^0}{1/3^0} = 1,$$

want 4 en 500 zijn beide niet deelbaar door 3.

Merk op dat met een p-adische bril op, een rationaal getal er heel *klein* uitziet als de teller *heel deelbaar* is door p (waarmee we bedoelen dat

[7] Het woord "p-adisch" bestaat uit twee delen, waarbij "p" staat voor een priemgetal. Het achtervoegsel "adisch" is vrij ongebruikelijk in de Nederlandse taal. Sterker nog, het wordt enkel en alleen gebruikt in deze wiskundige context. Als we werken met een concreet priemgetal, dan spreken we over "2-adisch", "3-adisch", "5-adisch", etc.

[8] Rationale getallen zijn getallen van de vorm $\frac{x}{y}$, waarbij x en y gehele getallen zijn en y niet gelijk is aan 0. We noemen x de *teller* en y de *noemer* van $\frac{x}{y}$. Voorbeelden van rationale getallen zijn $\frac{1}{2}$, $\frac{-2}{3}$, $\frac{5}{7}$ en $\frac{1}{2014}$. Voorbeelden van getallen die *geen* rationaal getal zijn, zijn $\sqrt{2}$ en π (beweren dat $\sqrt{2}$ geen rationaal getal is, kon je dood betekenen ten tijde van de Pythagoreërs).

de teller een heel grote macht van p bevat) en er heel *groot* uitziet als de noemer *heel deelbaar* is door p. Als noch de teller, noch de noemer deelbaar zijn door p, dan zal de p-adische absolute waarde gelijk zijn aan 1 (zie het laatste voorbeeld hierboven).

Bijvoorbeeld: 3^{1000} heeft een heel kleine 3-adische absolute waarde want $|3^{1000}|_3 = \frac{1}{3^{1000}}$ en $\frac{1}{3^{1000}}$ heeft een heel grote 3-adische absolute waarde want

$$\left|\frac{1}{3^{1000}}\right|_3 = \frac{|1|_3}{|3^{1000}|_3} = \frac{1/3^0}{1/3^{1000}} = 3^{1000}.$$

De getallen 1000 en $\frac{1}{1000}$ daarentegen, hebben 3-adische absolute waarde 1.

Zoals aangekondigd hierboven kunnen we nu een nieuwe afstand definiëren tussen twee rationale getallen: de *p-adische afstand* tussen x en y is per definitie gelijk aan $|x - y|_p$. Zoals we eerder al opmerkten is het voldoende om aan te tonen dat $|\ |_p$ een absolute waarde is om na te gaan dat dit een afstandsfunctie is, maar het bewijs hiervan zou ons te ver leiden in dit hoofdstuk.

Het bestuderen van meetkundige objecten (punten, rechten, oppervlakken, functies, etc.) gebruik makende van de p-adische afstand noemt men *p-adische meetkunde*. Dit is het centrale begrip in deze thesis.

De productformule

We hebben nu drie verschillende absolute waardes bestudeerd, namelijk de traditionele absolute waarde $|\ |$, de triviale absolute waarde $|\ |_{\mathrm{triv}}$ en de p-adische[9] absolute waarde $|\ |_p$. Elk van deze absolute waardes geeft aanleiding tot een andere afstand. Voor wie zich afvraagt waarom we zoveel moeite hebben gedaan om deze nieuwe absolute waardes in te voeren en vooral waarom we niet op zoek gaan naar nog meer absolute waardes, hebben we een leuke verassing in petto. Er is namelijk bewezen

[9] Eigenlijk hebben we voor elk priemgetal p een absolute waarde $|\ |_p$ gedefinieerd en er zijn oneindig veel priemgetallen, dus zijn er ook *oneindig veel* verschillende p-adische absolute waardes. Het is echter gebruikelijk om alle p-adische absolute waardes *samen* te beschouwen.

dat dit de enige drie mogelijk absolute waardes zijn die we *kunnen* definiëren op de rationale getallen.

We kennen dus *alle* absolute waardes die kunnen gedefinieerd worden op de rationale getallen. Met andere woorden, als je aan alle rationale getallen een bepaalde grootte wilt toekennen, dan kan dat essentieel op drie verschillende manieren: ofwel is het de traditionele absolute waarde $|\ |$, ofwel is het de triviale absolute waarde $|\ |_{\text{triv}}$, ofwel is het een p-adische absolute waarde $|\ |_p$ voor een bepaald priemgetal p. We sluiten dit hoofdstuk af met een formule die het verband geeft tussen die verschillende absolute waardes: de *productformule*. Voor elk rationaal getal x verschillend van nul geldt de volgende formule:

$$|x| \cdot \prod_{p \text{ priem}} |x|_p = 1,$$

waarbij het linkerlid moet gelezen worden als het product van de traditionele absolute waarde en *alle* p-adische absolute waardes, voor alle priemgetallen. Met andere woorden luidt de formule

$$|x| \cdot |x|_2 \cdot |x|_3 \cdot |x|_5 \cdots = 1.$$

Dit is een prachtige formule! Ze geeft niet alleen een verband tussen de traditionele absolute waarde en *alle* p-adische absolute waardes, maar doet dit ook op een enorm eenvoudige manier. Het rechterlid had immers eender wat kunnen zijn, maar het is eenvoudigweg 1. Dit is gewoon pure schoonheid.

Merk op dat hoewel de bovenstaande formule de gebruikelijke productformule is, het enigszins spijtig is dat de triviale absolute waarde er niet in voorkomt. Ook geldt de formule niet voor het getal 0, aangezien het linkerlid dan gelijk is aan 0. Hierdoor lijkt de volgende formule net nog iets mooier dan de klassieke productformule: voor *elk* rationaal getal x geldt dat

$$|x| \cdot \prod_{p \text{ priem}} |x|_p = |x|_{\text{triv}}.$$

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